Department of Mathematics
University of Illinois at Urbana-Champaign

MATH 417 – SPRING 2017 – SECTION B1
MIDTERM 2

MARCH 17, 2017

Midterm Duration: 50 m

SOLUTIONS

1. (25 points) Let $M_2(\mathbb{R})$ be the ring of real 2×2-matrices. Consider the subset $B \subset M_2(\mathbb{R})$ given by:

$$B = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

Which of the following statements are true/false (justify!):
(a) B is a subring of $M_2(\mathbb{R})$;
(b) B is an ideal of $M_2(\mathbb{R})$;
(c) B has no zero divisors.

Solution:

(a) True. The difference and the product of two matrices in B is a matrix in B:

$$\begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} - \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

where $a = a_1 - a_2$ and $b = b_1 - b_2$.

$$\begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

where $a = a_1a_2 - b_1b_2$ and $b = a_1b_2 + a_2b_1$.

(b) False. The product of a matrix in $M_2(\mathbb{R})$ by a matrix in B may fail to be in B, for example $I \in B$ but:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \notin B.$$

(c) True. Every non-zero matrix in B is invertible:

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^{-1} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in B.$$

2. (25 points) Let $A = \mathbb{R}[x]$ be the ring of polynomials with real coefficients and let $I \subset \mathbb{R}[x]$ be the subset:

$$I = \{a_2x^2 + a_3x^3 + \cdots + a_nx^n : n \geq 2, a_i \in \mathbb{R}\}.$$

Is it true that I is an ideal in $\mathbb{R}[x]$? Justify.

Solution:

I is an ideal since:

(1) it is closed for the difference (by adding zeros, we can assume the polynomials have the same degree):

$$(a_2x^2 + a_3x^3 + \cdots + a_nx^n) - (b_2x^2 + b_3x^3 + \cdots + b_nx^n) =$$
$$= (a_2 - b_2)x^2 + (a_3 - b_3)x^3 + \cdots + (a_n - b_n)x^m \in I.$$

(2) the product of any polynomial in I by any polynomial $p(x) = p_0 + p_1x + p_2x^2 + \cdots p_mx^m \in \mathbb{R}[x]$ is still a polynomial in I:

$$(a_2x^2 + a_3x^3 + \cdots + a_nx^n)(p_0 + p_1x + p_2x^2 + \cdots p_mx^m) =$$
$$= a_2p_0x^2 + \cdots \in I.$$

3. (25 points) Let $q \in \mathbb{H}$ be a quaternion:

$$q = a + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Show that $q^2 = -1$ if and only if q is purely imaginary and has norm 1:

$$q^2 = -1 \quad \Longleftrightarrow \quad q = -\bar{q},\ ||q|| = 1.$$

Hint: Recall that $||q||^2 = q\bar{q}$.

Solution:

Assume that $q^2 = -1$. Then:

$$1 = ||q^2|| = ||q||^2 \quad \Longrightarrow \quad ||q|| = 1.$$

and also:

$$q = -\frac{1}{q} = -\frac{\bar{q}}{q\bar{q}} = -\frac{\bar{q}}{||q||^2} = -\bar{q}.$$

Conversely, if $q = -\bar{q}$ and $||q|| = 1$ then:

$$q^2 = q(-\bar{q}) = -q\bar{q} = -||q||^2 = -1.$$

4. (25 points) $(A, +, \cdot)$ be an ordered ring with identity $1 \neq 0$. Show that A is not finite.

SOLUTION:

Let A^+ be the set of positive elements and assume, by contradiction, that A was finite. Then A^+ is also finite so it has a maximum: $a_0 = \max(A^+)$.

Now observe that, since $1 \neq 0$, we have $1 > 0$. But then:

$$a_0, 1 \in A^+ \quad \Longrightarrow \quad a_0 + 1 \in A^+,$$

and:

$$a_0 + 1 > a_0 + 0 = a_0,$$

contradicting the fact that a_0 was the maximum of A^+.

5. (Extra Credit: 10 points) Let $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be the map:

$$\boldsymbol{f}(x, y, z) = (y, -x, z) + (1, 1, 0).$$

Show that $\boldsymbol{f}$ is a symmetry of the set:

$$\Omega = \{(n_1, n_2, n_3) \in \mathbb{R}^3 : n_1, n_2, n_3 \in \mathbb{Z}\}.$$

SOLUTION:

Note that we can write $\boldsymbol{f}$ in the form:

$$\boldsymbol{f}(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{a},$$

where:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{a} = (1, 1, 0).$$

Since:

$$A^T A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I,$$

we see that A is an orthogonal matrix, so $\boldsymbol{f}$ is an isometry.

If $(n_1, n_2, n_3) \in \Omega$ then:

$$\boldsymbol{f}(n_1, n_2, n_3) = (n_1 + 1, 1 - n_1, n_3) \in \Omega.$$

Since $\boldsymbol{f}$ is an isometry and $\boldsymbol{f}(\Omega) \subset \Omega$, we conclude that $\boldsymbol{f}$ is a symmetry of Ω.