Department of Mathematics
University of Illinois at Urbana-Champaign

MATH 417 – SPRING 2017 – SECTION B1

FEBRUARY 10, 2017

Midterm Duration: 50 m

SOLUTIONS

1. (25 points) Consider the group $H = \{-1, 1\}$ with the usual multiplication of integers. Show that every element g in the group $G = H \times H$ satisfies $g^2 = 1$.

SOLUTION: The elements of G are pairs $g = (h_1, h_2)$ with $h_1, h_2 \in H$. The identity in G is $1 = (1, 1)$ and multiplication is component by component, hence:

$$g^2 = (h_1, h_2)(h_1, h_2) = (h_1^2, h_2^2) = (1, 1) = 1.$$

2. (25 points) Consider the permutation $\sigma \in S_7$ given by:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 4 & 1 & 2 & 5 & 3 \end{pmatrix}$$

i) Find a decomposition of σ into a product of disjoint cycles.
ii) Find two distinct decompositions of σ into a product of transpositions.

Solution:
i) $\sigma = (1, 7, 3, 4)(2, 6, 5)$.

ii) $\sigma = (1, 4)(1, 3)(1, 7)(2, 5)(2, 6) = (7, 1)(7, 4)(7, 3)(2, 5)(2, 6)$.

3. (25 points) Give an example of a ring $(A, +, \cdot)$ in each of the following classes (justify your answer):

(a) A is a field;
(b) A is an integral domain, but it is not a field;
(c) A satisfies the cancellation law, but it is not an integral domain;
(d) A does not satisfy the cancellation law.

SOLUTION:

(a) $A = \mathbb{R}$ is a field, since there is a identity, every element is invertible and multiplication is commutative.

(b) $A = \mathbb{Z}$ is an integral domain since it is commutative, has an identity, and the cancellation law holds, but it is not a field since there elements which are not invertible;

(c) $A = 2\mathbb{Z}$ satisfies the cancellation law, but it is not an integral domain since it has no identity;

(d) $A = M_2(\mathbb{R})$ does not satisfy the cancellation law since it has zero divisors:

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

4. (25 points) Let $A \neq \{0\}$ be an integral domain. Show that if $x \in A$ satisfies $x^2 = 1$ then either $x = 1$ or $x = -1$.

HINT: $(x-1)(x+1) = x^2 - 1$.

SOLUTION: Using the hint, we have:

$$x^2 = 1 \iff (x-1)(x+1) = 0.$$

Since A satisfies the cancellation law, it has no zero divisors. Hence, if a product equals zero, then at least one of the factors must be zero. It follows that:

$$x - 1 = 0 \quad \text{or} \quad x + 1 = 0$$

Hence, we conclude that:

$$x = 1 \quad \text{or} \quad x = -1$$

5. (Extra Credit: 10 points) Is the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ isomorphic to the group $K = \{1, i, -1, -i\}$ (with complex multiplication)? Justify your answer.

Solution: Observe that for G every element satisfies $g^2 = 1$ (see problem 1). On the other hand, in K the element $k = i$ satisfies $k^2 = -1 \neq 1$. If there is an isomorphism $\phi : G \to K$ we must have $\phi(g) = i$ for some $g \in G$. But then:

$$-1 = i^2 = \phi(g)\phi(g) = \phi(g^2) = \phi(1) = 1.$$

This is a contradiction, so there is no isomorphism $\phi : G \to K$.