# 1. Let $S$ be the shift operator on $\ell^\infty(\mathbb{N})$ defined by $S((x_n)) = (x_{n+1})$.

1. Show that the upper limit $p((x_n)) = \limsup x_n$ defines a sublinear functional on $\ell^\infty(\mathbb{N})$ which satisfies $p \circ S = p$.

2. Show that

$$q((x_n)) = \inf_{k \in \mathbb{N}} \left\{ \frac{p\left((x_n) + S((x_n)) + \cdots + S^{k-1}((x_n))\right)}{k} \right\}$$

is a sublinear functional on $\ell^\infty(\mathbb{N})$ such that $q((x_n)) \leq p((x_n))$ for all $(x_n) \in \ell^\infty(\mathbb{N})$.

**Proof of Part 1:** Given bounded sequences $(x_n)$ and $(y_n)$, we have

$$\sup\{x_k + y_k, \cdots, x_n + y_n, \cdots\} \leq \sup\{x_k, \cdots, x_n, \cdots\} + \sup\{y_k, \cdots, y_n, \cdots\}$$

for all $k \in \mathbb{N}$. From this we can conclude that

$$p((x_n + y_n)) = \limsup (x_n + y_n) \leq \limsup x_n + \limsup y_n = p((x_n)) + p((y_n)).$$

It is easy to verify that for any $a \geq 0$, we have $p(a(x_n)) = ap((x_n))$. Therefore, $p$ is a sublinear functional on $\ell^\infty(\mathbb{N})$. We have $p \circ S = p$ by definition of upper limit.

**Proof of Part 2:** For arbitrary $\varepsilon > 0$, there exist positive integers $k$ and $l$ such that

$$\frac{p\left((x_n) + S((x_n)) + \cdots + S^{k-1}((x_n))\right)}{k} < q((x_n)) + \frac{\varepsilon}{2}$$

and

$$\frac{p\left((y_n) + S((y_n)) + \cdots + S^{l-1}((y_n))\right)}{l} < q((y_n)) + \frac{\varepsilon}{2}.$$

Then we have

$$q((x_n) + (y_n)) \leq \frac{p\left((x_n + y_n) + S((x_n + y_n)) + \cdots + S^{kl-1}((x_n + y_n))\right)}{kl} \leq \frac{p\left((x_n) + S((x_n)) + \cdots + S^{kl-1}((x_n))\right)}{kl} + \frac{p\left((y_n) + S((y_n)) + \cdots + S^{kl-1}((y_n))\right)}{kl} \leq \frac{p\left((x_n) + S((x_n)) + \cdots + S^{l-1}((x_n))\right)}{l} + \frac{p\left((y_n) + S((y_n)) + \cdots + S^{l-1}((y_n))\right)}{l} < q((x_n)) + q((y_n)) + \varepsilon,$$

where we used the fact that

$$p\left((x_n) + \cdots + S^{kl-1}((x_n))\right) \leq p\left((x_n) + \cdots + S^{kl-1}((x_n))\right) + \cdots + p\left(S^{k(l-1)}((x_n)) + \cdots + S^{k-1}((x_n))\right) = l \times p\left((x_n) + \cdots + S^{k-1}((x_n))\right).$$

It is easy to show that for $a \geq 0$, $q(a(x_n)) = aq((x_n))$. So $q$ is a sublinear functional on $\ell^\infty(\mathbb{N})$. Since $p$ is sublinear and $p \circ S = p$, it is easy to see that $q((x_n)) \leq p((x_n))$ for all $(x_n) \in \ell^\infty(\mathbb{N})$. 
\# 2. Show that there exists a positive linear functional $F \in \ell^\infty(\mathbb{N})^*$ of norm $\|F\| = 1$ such that

1. $\lim x_n \leq F((x_n)) \leq \lim x_n$,
2. $F((x_n)) = F(S(x_n))$ for all $(x_n) \in \ell^\infty(\mathbb{N})$.

We call $F((x_n))$ the Banach limit of $(x_n)$ and write $F((x_n)) = \text{LIM } x_n$.

**Proof:** Let us consider the linear functional $f : c(\mathbb{N}) \to \mathbb{R}$ given by $f((x_n)) = \lim x_n$. In this case, it is easy to see that for any $(x_n) \in c(\mathbb{N})$,

$$f((x_n)) = \lim x_n = p((x_n)) = q((x_n)).$$

Then we can apply the Hahn-Banach extension theorem to get a linear functional $F : \ell^\infty(\mathbb{N}) \to \mathbb{R}$ such that

$$F((x_n)) \leq q((x_n)) \leq p((x_n)) = \lim x_n.$$  

We also have

$$-F((x_n)) = F((-x_n)) \leq q((-x_n)) \leq p((-x_n)) = \lim(-x_n) = -\lim x_n.$$  

It follows that

$$\lim x_n \leq F((x_n)) \leq q((x_n)) \leq p((x_n)) = \lim x_n.$$  

It is clear that $\|F\| = 1$. This proves Part 1. To prove Part 2, we notice that

$$q((x_n) - S((x_n))) \leq \frac{p((x_n) - S_k((x_n)))}{k} = \frac{p((x_n)) + p(S_k(-x_n))}{k} \to 0.$$  

This shows that for any $(x_n) \in \ell^\infty(\mathbb{N})$, we have

$$F((x_n)) - F(S((x_n))) = F((x_n) - S((x_n))) \leq q((x_n) - S((x_n))) \leq 0.$$  

Hence we have $F((x_n)) \leq F(S((x_n)))$. Apply the negative sign, we get

$$-F((x_n)) = F((-x_n)) \leq F(S((-x_n))) = -F(S((x_n))).$$  

Hence we also have $F(S((x_n))) \leq F((x_n))$. This proves Part 2.

\# 3. Let $X$ be a compact Hausdorff space. If $\mathfrak{F}$ is a cone in $C(X) = C(X, \mathbb{R})$ such that for every $f \in \mathfrak{F}$, there exists a point $x_0 \in X$ such that $f(x_0) \geq 0$, show that there exists a positive Radon measure $\mu \in M(X)$ such that $\int_X f d\mu \geq 0$ for all $f \in \mathfrak{F}$.  

**Proof:** Let us consider the set $A = \{ f \in C(X) : f(x) < 0 \text{ for all } x \in X \}$. It is easy to see that $A$ is an open convex set in $C(X)$ such that $A \cap \mathfrak{F} = \emptyset$ and if $f \in A$ then $tf \in A$ for all $t > 0$. We can apply the Hahn-Banach separation theorem to find a linear functional $F$ in $C(X)$ and a real number $r$ such that

$$F(A) < r \leq F(\mathfrak{F}).$$

Since $\mathfrak{F}$ is a cone in $C(X)$, we can conclude that $r \leq 0$. On the other hand, if $f \in A$, then $tf \in A$ and thus $tF(f) < r$ for all $t > 0$. Now letting $t \to 0$, we get $r \geq 0$. This shows that $r = 0$ and $F(f) < 0$ for all negative function $f \in A$. Therefore $F$ must be a positive linear functional on $C(X)$ such that $F(f) \geq 0$ for all $f \in \mathfrak{F}$. Using Radon theorem, we can obtain a positive Radon measure $\mu$ on $(X, \mathcal{B})$ such that $\int_X f d\mu = F(f) \geq 0$ for all $f \in \mathfrak{F}$. This completes the proof.
# 4. Let $T : V \to W$ be a bounded linear map from a Banach space $V$ into a Banach space $W$.

1. Show that $\ker T = \{ x \in V : T(x) = 0 \}$, the kernel of $T$, is a closed subspace of $V$.

2. Let $\pi : V \to V/\ker T$ be the induced quotient map. Show that there exists a unique bounded linear map $\tilde{T} : V/\ker T \to W$ such that $\| \tilde{T} \| = \| T \|$ and $T = \tilde{T} \circ \pi$.

**Proof of Part 1:** Suppose that $x_n \in \ker T$ and $x_n \to x \in V$. Then the continuity of $T$ implies that $T(x) = T(\lim x_n) = \lim T(x_n) = 0$. This shows that $\ker T$ is a closed subspace of $V$.

**Proof of Part 2:** We can define a map $\tilde{T} : V/\ker T \to W$ given by $\tilde{T}([x]) = T(x)$. This map is well-defined since if $[x] = [x']$, i.e. $x - x' \in \ker T$, we have $T([x']) = T(x') = T(x) = \tilde{T}([x])$. It is easy to see that this is a linear map. For any $[x] \in V/\ker T$, we have $[x] = [x + y]$ for all $y \in \ker T$. It follows that

$$\| \tilde{T}([x]) \| = \| \tilde{T}([x + y]) \| = \| T(x + y) \| \leq \| T \| \| x + y \|.$$ 

Taking the infimum for $y \in \ker T$, we get $\| \tilde{T} \| \leq \| T \|$.

On the other hand, for any $\varepsilon > 0$, there exists $x \in V$ such that $\| x \| < 1$ and $\| T \| - \varepsilon < \| \tilde{T}(x) \| = \| \tilde{T}([x]) \| \leq \| T \|$. This shows that $\| T \| \leq \| \tilde{T} \|$. Therefore, we have $T = \tilde{T} \circ \pi$ with $\| \tilde{T} \| = \| T \|$. Finally, if we have another map $\hat{S} : V/\ker T \to W$ such that $\hat{S} \circ \pi = T = \tilde{T} \circ \pi$, then we must have $\hat{S} = \tilde{T}$ since for all $[x] \in V/\ker T$,

$$\hat{S}([x]) = \hat{S} \circ \pi(x) = T(x) = \tilde{T} \circ \pi(x) = \tilde{T}([x]).$$

# 5. Let $T : V \to W$ be a linear map from a Banach space $V$ into a Banach space $W$. If $f \circ T \in V^*$ for every $f \in W^*$, show that $T$ is bounded on $V$.

**Proof by Closed Graph Theorem:** Suppose that $x_n \to x$ in $V$ and $T(x_n) \to y$ in $W$. Since $f \circ T = T^*(f) \in V^*$ for arbitrary $f \in W^*$, we get

$$f(y) = f(\lim T(x_n)) = \lim f(T(x_n)) = \lim (f \circ T)(x_n) = (f \circ T)(\lim x_n) = (f \circ T)(x) = f(T(x)).$$

This shows that $T(x) = y$, i.e. $T$ has the closed graph. Therefore, $T$ must be a bounded linear map on $V$ by the Closed Graph theorem.

**Proof by Uniformly Bounded Principle:** Let us first consider an index set $I = B(0,1)^-$, the closed unit ball of $V$. Then $\{ T(x) \}_{x \in I}$ is a subset of $W$.

Consider the canonical embedding $\wedge : W \to W^{**}$, we can regard $\{ \tilde{T}(x) \}_{x \in I}$ as a subset in $W^{**}$. This gives a family of bounded linear functionals $\tilde{T}(x) : W^* \to \mathbb{R}$ on $W^*$. Now for each $f \in W^*$, $T^*(f) = f \circ T$ is a bounded linear functional in $V^*$. Therefore,

$$\{ \tilde{T}(x)(f) \}_{x \in I} = \{ f(T(x)) \}_{x \in I} = \{ (f \circ T)(x) \}_{x \in I}$$

is a bounded set in $\mathbb{R}$. Since $W^*$ is a Banach space, then the Uniform Bounded Principle implies that $\{ \tilde{T}(x)(f) \}_{x \in I}$ is uniformly bounded, i.e. there exists some $C > 0$ such that $\| T(x) \| = \| \tilde{T}(x) \| \leq C < \infty$ for all $x \in I = B(0,1)^-$. This shows that $T$ is a bounded linear map on $V$. 
