Math 347, 2nd Exam, Review and Practice Problems

Part I sup and inf:

1. Recall the definition of upper bound, least upper bound, lower bound, greatest lower bound, maximum element, and minimum element of S.

2. Recall "Completeness Axiom of Real Numbers", "Archimedean Property", Corollaries and "Density of Rational Numbers" / "Density of Irrational Numbers".

3. Determine whether the following sets are bounded above (respectively, bounded below). If yes, find the sup (respectively, find the inf). You should be able to verify your answer by using definition and the above properties.

   Let $S = \{x \in \mathbb{R} : x^2 > 2x + 8\}$, $T = \{1 - \frac{1}{n}\}$, $K = [-\pi, \pi] \cap Q$, and $L = \{x_n\}$ is the sequence such that $x_1 = 6$, and $x_{n+1} = \frac{1}{2}x_n + 1$ for all $n \geq 1$.

4. Let (a_n) be a bounded sequence of real numbers. Show that

   $$\sup\{a + a_n\} \leq a + \sup\{a_n\}$$

   Show that in fact we have

   $$\sup\{a + a_n\} = a + \sup\{a_n\}$$

5. Example: Let $S = \{x \text{ irrational} : x^2 \leq 1\}$. Find $\inf(S)$ and $\sup(S)$.
   Is $\inf(S)$ (resp, $\sup(S)$) a minimum (resp, a maximum) element in S ?
   Show that there exists a sequence $x_n \in S$ which converges to $\sup(S)$.
   Show that there exists a sequence $x_1 < x_2 < \cdots < x_n < \cdots$ in S such that $\{x_n\}$ converges to $\sup(S)$.

6. Nested intervals (see HW problems).

7. Review all homework assignments.

Part II Sequences:

1. Recall the definition of $\lim x_n = x$. Use the definition to show that

$$\lim \frac{2n-1}{3n+1} = \frac{2}{3} \text{ and } \lim \frac{2n+1}{2n-5} = 1.$$

2. Recall the definition of $\lim x_n \neq x$. Use the definition to show that $\frac{(-2)^n+1}{2^n-1}$ does not converges to 1, -1.

   Actually this is a divergent sequence, i.e. it does not converge to any real number.

3. Recall some properties of sequences:

   - Every convergent sequence must be bounded.
     Therefore, an unbounded sequence must diverge.
   - Bounded sequence needs not converge.
   - If $\{x_n\}$ converges to x, then every subsequence must converges to x.
   - Therefore, if a sequence $\{x_n\}$ has two subsequences converging to different numbers, then (x_n) must be divergent.

4. If (x_n) is monotone increasing and bounded above, then limit exists and

$$\lim x_n = \sup\{x_n\}.$$

   If (x_n) is monotone decreasing and bounded below, then limit exists and

$$\lim x_n = \inf\{x_n\}.$$

   Find limit of some monotone sequences.

6. Review all basic definitions, limit, (i.e. convergence and divergence), and some fundamental theorems like Bolzano-Weiestrass theorem.

5. Review all homework assignments.