Math 347, 2nd Exam for Section D2, Solution

Monday, April 2, 2018, 11:00-11:50am

This is a close book exam. Do all 4 problems.

I (20 points) Let S be a non-empty bounded subset of real numbers.

1. State the definition of $\sup(S)$, describing condition 1) and condition 2).

   Definition:. A real number M is the least upper bound of S if

   1) M is an upper bound of S, i.e. $s \leq M$ for all $s \in S$,

   2) M is the least upper bound of S, i.e. if M' is an upper bound of S, then $M \leq M'$.

2. Use the definition to show that for any positive real number $a > 0$,

   $$\sup(a \cdot S) = a \cdot \sup(S),$$

   where we let $a \cdot S = \{a \cdot s : s \in S\}$.

   Proof: Let $M = \sup(S)$. We want to show that $aM = \sup(aS)$.
   Verify condition 1: For any $s \in S$, we have $s \leq M$ and since $a > 0$, we get $as \leq aM$ for all $s \in S$. This shows that aM is an upper bound of aS.
   Verify condition 2: Now suppose M' is an upper bound of aS, we have $as \leq M'$ and thus $s \leq \frac{M'}{a}$ for all $s \in S$. This shows that $\frac{M'}{a}$ is an upper bound of S. Therefore, we have $M \leq \frac{M'}{a}$ or equivalently, $aM \leq M'$. This shows that

   $$\sup(aS) = aM = a \sup(S).$$

II (30 points) Let $S = \{x \text{ irrational} : 0 \leq x \leq 2\}$.

1. Use the definition to show that $\sup(S) = 2$.

   Proof: (1) It is clear that 2 is an upper bound of S.
   (2') For any $0 \leq u < 2$, it is known from the "Density of Irrational Numbers" that there exists an irrational number q such that $u < q < 2$. This number q belongs to S and this shows that u is not an upper bound of S. Therefore, we have $\sup(S) = 2$ by condition (1) and condition (2').

2. Does S has any maximum element ? Explain your answer.

   Answer: The set S does not have any maximum element.
   Suppose S has a maximum element s_0. As we have discussed in class that s_0 is the least upper bound of S, i.e. $s_0 = \sup(S) = 2$. But 2 is not contained in S. This contradiction shows that S has no maximum element.

3. Show that there exists a sequence $x_n \in S$ such that (x_n) converges to $\sup(S) = 2$.

   Proof: Since $2 = \sup(S)$ is the least upper bound of S, for any $n \in \mathbb{N}$, $2 - \frac{1}{n}$ is not an upper bound of S. Hence there exists an element, say $x_n \in S$, such that
   $$2 - \frac{1}{n} < x_n \leq 2.$$
   Then $\{x_n\}$ is a sequence contained in S and $\{x_n\}$ converges to $\sup(S) = 2$ by the Squeeze theorem.

   A Direct Construction: You may construct a sequence as follows.
   For each $n \in \mathbb{N}$, $x_n = 2 - \frac{\sqrt{2}}{n}$ is an irrational number contained in $[0, 2]$. Therefore $\{x_n\}$ is a sequence in S converging to $2 = \sup(S)$.

III (25 points)

1. Describe the definition that a sequence $\{x_n\}$ does not converge to a real number x.

   Definition: There exists an $\varepsilon_0 > 0$ such that for all $k \in \mathbb{N}$, there exists $n_k \geq k$ such that
   $$|x_{n_k} - x| \geq \varepsilon_0.$$

2. Use the definition to show that the sequence $x_n = \frac{2n+1}{2n-1}$ does not converge to 0.

   Proof: There exists an $\varepsilon_0 = 1 > 0$ such that for all $k \in \mathbb{N}$, there exists $n_k = k$ such that
   $$|x_{n_k} - 0| = \left|\frac{2k+1}{2k-1}\right| = 1 + \frac{2}{2k-1} \geq \varepsilon_0 = 1.$$
   This shows that $\{x_n\}$ does not converge to 0.

IV (25 points) Answer whether the following statements are True or False. If true because of some theorem, cite the theorem, and if false, provide a counter-example.

1. Let S be a non-empty subset S of $\mathbb{R}$. If S has an upper bounded, then S must have a least upper bound.

   True. This statement is known as the "Completeness Axiom of $\mathbb{R}$".

2. Let S be a bounded non-empty subset of $\mathbb{R}$. Then $\sup(S)$ must be a maximum element in S.

   False. Look at open interval $S = (0, 1)$. It is a bounded non-empty subset of $\mathbb{R}$, but $\sup(S) = 1$ is not a (maximum) element in S.

3. Every monotone increasing sequence $\{x_n\}$ must converge to some real number x, (i.e. $\lim x_n = x$ for some real number x).

   False. Without boundedness, the statement is false. Look at $\{x_n\} = \{n\}$. This sequence is monotone increasing, but unbounded above. It does not converge to any real number.

4. For arbitrary irrational numbers $x < y$, there exists a rational number r such that

   $$x < r < y.$$

   True. This statement is true by the "Density of Rational Numbers".

5. For any real number x, there exists a positive integer $n \in \mathbb{N}$ such that $x < n$.

   True. This statement is known as the "Achimedean Property".