Math 347, 2nd Exam for Section B1, Solution

Monday, April 2, 2018, 9:00-9:50am

This is a close book exam. Do all 4 problems.

I (20 points) Let S be a non-empty bounded subset of real numbers.

1. State the definition of $\inf(S)$, describing condition 1) and condition 2).

   Definition: m is the greatest lower bound, $\inf(S)$, of S if

   (1) m is a lower bound of S, i.e. $m \leq s$ for all $s \in S$.

   (2) For any lower bound m' of S, we have $m' \leq m$.

2. Use the definition to show that for any real number a,

$$\inf(a + S) = a + \inf(S),$$

   where we let $a + S = \{a + s : s \in S\}$.

   Proof: Let $m = \inf(S)$. We want to show $a + m = \inf(a + S)$.

   (1) For any $s \in S$, we have $m \leq s$ and thus $a + m \leq a + s$. This shows that $a + m$ is a lower bound of $a + S$.

   (2) If m' is a lower bound of $a + S$, we have $m' \leq a + s$ for all $s \in S$. Therefore, we have $m' - a \leq s$ for all $s \in S$. So $m' - a$ is a lower bound of S and thus $m' - a \leq m$. This shows that $m' \leq a + m$.

   Therefore, we can conclude from (1) and (2) that $a + m$ is the greatest lower bound of $a + S$.

II (30 points) Let $S = \{x \text{ irrational} : 0 \leq x \leq 2\}$.

1. Use the definition to show that $\inf(S) = 0$.

   Proof: (1) It is clear that 0 is a lower bound of S.

   (2') For any $0 < u \leq 2$, it is known from the "Density of Irrational Numbers" that there exists an irrational number q such that $0 < q < u \leq 2$. Then this number q belongs to S and this shows that u is not a lower bound of S. Therefore by condition (1) and condition (2'), 0 is the greatest lower bound of S, i.e., $0 = \inf(S)$.

2. Does S have any minimum element ? Explain your answer.

   Answer: The set S does not have any minimum element.
   Suppose that S has a minimum element s_0. Then s_0 must be the greatest lower bound of S and thus we must have $s_0 = \inf(S) = 0$. However, 0 is not an element in S. This contradiction shows that S does not has any minimum element.

3. Show that there exists a sequence $x_n \in S$ such that (x_n) converges to $\inf(S) = 0$.

   Proof: Since $\inf(S) = 0$, for each $n \in \mathbb{N}$, $0 < \frac{1}{n}$ is not an upper bound of S. Therefore, there exists an element, say $x_n \in S$, such that

   $$\inf(S) = 0 < x_n < \frac{1}{n}.$$

   By Squeeze theorem, $\lim x_n = \lim \frac{1}{n} = 0 = \inf(S)$.

   Remark: We should give students full credit if they take the following approach.

   Proof: There exists a sequence $x_n = \frac{\sqrt{2}}{n+1}$ of irrational numbers in S such that $\lim x_n = 0 = \inf(S)$.

III (25 points)

1. Describe ε-definition $\lim x_n = x$.

   Definition: A sequence (x_n) converges to a real number x if for arbitrary $\varepsilon > 0$, there exists a positive integer $K \in \mathbb{N}$ such that for all $n \geq K$, we have $|x_n - x| < \varepsilon$.

2. Use the above definition to show that $\lim \frac{2n+1}{2n-1} = 1$.

   Discussion: We want to show that for any $\varepsilon > 0$, there exists a positive integer $K \in \mathbb{N}$ such that for all $n \geq K$

   $$\left|\frac{2n+1}{2n-1} - 1\right| = \frac{|(2n+1)-(2n-1)|}{|2n-1|} = \frac{2}{n+(n-1)} \leq \frac{2}{n} < \varepsilon.$$

   Hence we can take $K > \frac{2}{\varepsilon}$.

   Proof: For arbitrary $\varepsilon > 0$, there. exists a positive integer $K > \frac{2}{\varepsilon}$ such that for all $n \geq K$, we have

   $$\left|\frac{2n+1}{2n-1} - 1\right| = \frac{|(2n+1)-(2n-1)|}{|2n-1|} = \frac{2}{n+(n-1)} \leq \frac{2}{n} \leq \frac{2}{K} < \varepsilon.$$

   This shows that $\lim \frac{2n+1}{2n-11} = 1$.

IV (25 points) Answer whether the following statements are True or False. If true because of some theorem, cite the theorem, and if false, provide a counter-example.

1. Let S be a non-empty subset S of $\mathbb{R}$. If S has an upper bounded, then S must have a least upper bound.

   True. This statement is known as the "Completeness Axiom of $\mathbb{R}$".

2. Let S be a bounded non-empty subset of $\mathbb{R}$. Then $\sup(S)$ must be a maximum element in S.

   False. Look at open interval $S = (0, 1)$. It is a bounded non-empty subset of $\mathbb{R}$, but $\sup(S) = 1$ is not a (maximum) element in S.

3. Every monotone increasing sequence $\{x_n\}$ must converge to some real number x, (i.e. $\lim x_n = x$ for some real number x).

   False. Without boundedness, the statement is false. Look at $\{x_n\} = \{n\}$. This sequence is monotone increasing, but unbounded above. It does not converge to any real number.

4. For arbitrary irrational numbers $x < y$, there exists a rational number r such that

   $$x < r < y.$$

   True. This statement is true by the "Density of Rational Numbers".

5. For any real number x, there exists a positive integer $n \in \mathbb{N}$ such that $x < n$.

   True. This statement is known as the "Achimedean Property".