Math 347, 1st Exam Solution

Monday February 19, 2018

This is a close book exam. Do all 4 problems.

I (25 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by $f(x) = \ln(x^2 + 1)$.

(1) Determine the image $f(\mathbb{R})$ of f.
Solution: $f(\mathbb{R}) = [0, \infty)$.

(2) Is f an injection ? Justify your answer ?
Answer: No. There exists $x = 1, or - 1$ such that $f(-1) = f(1) = \ln 2$.

(3) Find a subset A of $\mathbb{R}$ on which f defines a bijection from A onto $f(\mathbb{R})$.
Solution: We can chose $A = [0, \infty)$.

(4) Find the inverse function of f, i.e. find a function $g : f(\mathbb{R}) \to A$ such that $g \circ f(x) = x$ for all $x \in A$, and $f \circ g(y) = y$ for all $y \in f(\mathbb{R})$.
Solution: Let $y = \ln(x^2 + 1)$. We get $x = \sqrt{e^y - 1}$ and claim that $g(y) = \sqrt{e^y - 1}$ is the inverse function of f.

It is easy to verify that
$g(f(x)) = g(\ln(x^2 + 1)) = \sqrt{e^{\ln(x^2+1)} - 1} = \sqrt{(x^2 + 1) - 1} = x$
for all $x \in A$, and
$f(g(y)) = \ln((\sqrt{e^y - 1})^2 + 1) = \ln(e^y - 1 + 1) = y$
for all $y \in f(\mathbb{R})$. This shows that g is the inverse of f.

Remark: We may also consider $A^- = (-\infty, 0]$. In this case, we get $g(y) = -\sqrt{e^y - 1}$.

II (25 points)

(1) Find the negation of the following statement:
"There exists a real number $M \in \mathbb{R}$ such that for every $x \in \mathbb{R}$ there exists $y > x$ such that $f(y) < M$."
Solution: "For every real number $M \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that for all $y > x$ we have $f(y) \geq M$."

(2) Find the negation of the following statement:
" For every $\varepsilon > 0$, there exists a positive integer K such that for all $n \geq K$, we have $|a_n - a| < \varepsilon$."
Solution: "There exists an $\varepsilon > 0$ such that for all positive integer $K \in \mathbb{N}$, there exists an $n_K \geq K$ such that $|a_{n_K} - a| \geq \varepsilon$."

(3) Prove that for any positive integer $n \in \mathbb{N}$, $a = (n + \sqrt{2})^{\frac{1}{n}}$ is an irrational number.
Proof: Suppose the above statement is false. Then there exist some positive integer n such that $a = (n + \sqrt{2})^{\frac{1}{n}}$ is a rational number. Taking the n^{th} power, we get $\sqrt{2} = a^n - n$. This is a contradiction since $\sqrt{2}$ is irrational, but $a^n - n$ is rational. Therefore, for any positive integer $n \in \mathbb{N}$, $a = (n + \sqrt{2})^{\frac{1}{n}}$ is irrational.

III (20 points) Let $P_1, P_2, \cdots, P_n, \cdots$ be a sequence of statements.

(1) State the Strong Mathematics Induction.

Solution: First we need to verify that
⟨1⟩ For $n = 1$, the initial statement P_1 is true.
⟨2⟩ For $n \geq 2$, if $P_1, \cdots P_{n-1}$ are true, then P_n is true.
Then Strong Mathematical Induction tells us that all statements $\{P_n\}$ are true.

(2) Let (a_n) be a sequence of positive integers. If a_1 and a_2 are even, and $a_n = a_{n-1} + 3a_{n-2}$ for all $n \geq 3$. Prove that a_n are even for all $n \in \mathbb{N}$.

Proof: Initial Step: It is true that a_1 and a_2 are even (this is given).

Induction Step: For $k \geq 3$, suppose $a_1, \cdots, a_{k-2}, a_{k-1}$ are even, we can write $a_{k-1} = 2m$ and $a_{k-2} = 2l$ for some $m, l \in \mathbb{N}$. It follows that $a_k = a_{k-1} + 3a_{k-2} = 2(m + 3l)$ is even. So by Strong Mathematical Induction, all statements $\{P_1, \cdots P_n, \cdots)\}$ are true.

IV (30 points) Let $f : A \to B$ be a function.

(1) State the following definitions

(a) f is an injection.
Definition: A function $f : A \to B$ is an injection if for each $b \in B$ there is at most one $a \in A$ such that $b = f(a)$.

(b) f is a surjection.
Definition: A function $f : A \to B$ is a surjection if for each $b \in B$ there is at least one $a \in A$ such that $b = f(a)$.

(2) Let $g : B \to A$ be a function such that $g \circ f(a) = a$ for all $a \in A$.

(a) Is $f : A \to B$ a surjection ? If yes, prove it. If no, provide an example.
Answer: No. f is not necessarily a surjection. We may consider the following functions
$f : \{1\} \to \{1, 2\}$ with $f(1) = 1$, and $g : \{1, 2\} \to \{1\}$ with $g(1) = g(2) = 1$. It is clear that $g \circ f : A \to A$ is a bijection with $g \circ f(1) = 1$. But f is not a surjection from A onto B.

(b) Is $g : B \to A$ a surjection ? If yes, prove it. If no, provide an example.
Answer: Yes. For any $a \in A$, there exists $b = f(a) \in B$ such that $g(b) = g(f(a)) = a$ This shows that $g : B \to A$ is a surjection.