Math347, Homework #9, Solution
Due to Wednesday, April 11, 2018

Page 134, #6.8 a). Use the Euclidean algorithm to compute $gcd(224, 126)$, and express $gcd(224, 126)$ as the integer combination of 224 and 126.

Since $224 = 126 + 98 \Leftrightarrow 98 = 224 - 126$, we get $gcd(224, 126) = gcd(126, 98)$.
Since $126 = 98 + 28 \Leftrightarrow 28 = 126 - 98$, we get $gcd(126, 98) = gcd(98, 28)$.
Since $98 = 3\times 28 + 14 \Leftrightarrow 14 = 98 - 3\times 28$, we get $gcd(98, 28) = gcd(28, 14) = 14$.
This shows that $gcd(224, 126) = 14$ and we have

$$\begin{aligned} 14 &= 98 - 3 \times 28 = 98 - 3 \times (126 - 98) = -3 \times 126 + 4 \times 98 \\ &= -3 \times 126 + 4 \times (224 - 126) = 4 \times 224 - 7 \times 126. \end{aligned}$$

Page 134, #6.8 b). Use the Euclidean algorithm to compute $gcd(221, 299)$, and express $gcd(221, 229)$ as the integer combination of 221 and 229.

Since $299 = 221 + 78 \Leftrightarrow 78 = 299 - 221$, we get $gcd(299, 221) = gcd(221, 78)$.
Since $221 = 2 \times 78 + 65 \Leftrightarrow 65 = 221 - 2 \times 78$, we get $gcd(221, 78) = gcd(78, 65)$.
Since $78 = 65 + 13 \Leftrightarrow 13 = 78 - 65$, we get $gcd(78, 65) = gcd(65, 13) = 13$.
Therefore, we get $gcd(299, 221) = 13$ and

$$\begin{aligned} 13 &= 78 - 65 = 78 - (221 - 2 \times 78) = -221 + 3 \times 78 \\ &= -221 + 3 \times (299 - 221) = 3 \times 299 - 4 \times 221. \end{aligned}$$

Page 134, #6.9 b). Find all solution of the Diophatine equation

$$21x + 15y = 93.$$

Solution: Since $gcd(21, 15) = 3$, and 93 is a multiple of 3. So we have integers solutions. To simply the problem, we can divide the original equation by its greatest common divisor 3 , and consider all integer solutions of

$$7x + 5y = 31.$$

In this case, 7 and 5 are relatively prime. We can get one solution $(3, 2)$, and the general solution can be given by $(3 + 5k, 2 - 7k)$ for all $k \in \mathbb{Z}$.

Page 134, # 6.9 c). If exist, find all solution of the diophatine equation

$$60x + 42y = 104.$$

Solution: Since $gcd(60, 42) = 6$ and $104 = 16 \times 6 + 2$ is not a multiple of 6. So there is no integer solution for $60x + 42y = 104$.

Page 134, #6.17. Prove that $gcd(a + b, a - b) = gcd(2a, a - b) = gcd(a + b, 2b)$.

Proof: We have shown in class that for any positive integers a and b we have

$$gcd(a, b) = gcd(a - kb, b)$$

for any $k \in \mathbb{Z}$. We also note that

$$gcd(a, b) = gcd(\pm a, \pm b).$$

Therefore, we can conclude that

$$gcd(a + b, a - b) = gcd((a + b) + (a - b), a - b) = gcd(2a, a - b)$$

and

$$gcd(a + b, a - b) = gcd((a + b), (a - b) - (a + b)) = gcd(a + b, -2b) = gcd(a + b, 2b).$$

Page 134, #6.46. Find all integer solutions to $70x + 28y = 518$. Determine how many solutions have both variables positive.

Solution: Since $gcd(70, 28) = 14$ and 14 divides 518, we have integer solutions, and we can reduce this Diophantine equation to

$$5x + 2y = 37.$$

It is easy to see that $(7, 1)$ is a solution for $5x+2y = 27$. Since 5 and 2 are relatively prime, the general solutions are $(7 - 2k, 1 + 5k)$ for all $k \in \mathbb{Z}$.
All positive solutions are $(7, 1), (5, 6), (3, 11)$ and $(1, 16)$.

Page 134, #6.47. Find all integer solutions to $\frac{1}{60} = \frac{x}{5} + \frac{y}{12}$.

Solution: This is equivalent to find all integer solutuions of $12x+5y = 1$. Since 12 and 5 are relatively prime, we have integer solutions. It is easy to see that $(-2, 5)$ is a solution. The general solutions are $(-2 + 5k, 5 - 12k)$ for all $k \in \mathbb{Z}$.