Math347, Spring 2018, Homework #8 Solution
Due to Wednesday, March 28, 2018

HW 8.1. Let $\{x_n\}$ be a sequence of real numbers such that $\frac{-1}{n} \leq x_n \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Use Squeeze theorem to determine $\lim_{n\to\infty} \mathrm{e}^{x_n}$.

Solution: Since the exponential function e^x is monotone increasing on $\mathbb{R}$, we have

$$\mathrm{e}^{-\frac{1}{n}} \leq \mathrm{e}^{x_n} \leq \mathrm{e}^{\frac{1}{n}}$$

for all $n \in \mathbb{N}$. It is easy to see that

$$\lim \mathrm{e}^{-\frac{1}{n}} = \mathrm{e}^{-\lim \frac{1}{n}} = 1 = \mathrm{e}^{\lim \frac{1}{n}} = \lim \mathrm{e}^{\frac{1}{n}}.$$

By Squeeze theorem, we can conclude that $\lim \mathrm{e}^{x_n} = 1$.

HW 8.2. Let $\{x_n\}$ be a sequence of real numbers with $x_1 = 2$ and $x_{n+1} = 2 - \frac{1}{x_n}$ for all $n \in \mathbb{N}$. Show that $\{x_n\}$ is bounded and monotone decreasing. Find the limit if it exsists

Proof: (1) We claim that $1 < x_n \leq 2$ for all $n \in \mathbb{N}$. Let us use math induction to prove this claim. This is true for $n = 1$ (since $a_1 = 2$). Suppose for $k \geq 1$, we have $1 < x_k \leq 2$. Then $\frac{1}{2} \leq \frac{1}{x_k} < 1$ and thus $1 < x_{k+1} = 2 - \frac{1}{x_k} \leq 2$. By Mathematical Induction, this claim is true for all x_n.

(2) Consider $x_n - x_{n+1} = x_n - 2 + \frac{1}{x_n} = \frac{x_n^2 - 2x_n + 1}{x_n} = \frac{(x_n-1)^2}{x_n} > 0$. Therefore, $x_n > x_{n+1}$. The sequence is monotone decreasing.

(3) According to (1) and (2), the sequence $\{x_n\}$ is monotone decreasing and bounded below. Therefore by Monotone Convergence theorem, it is a convergent sequence. Let $x = \lim x_n$ be the limit of $\{x_n\}$. Taking limit for $x_{n+1} = 2 - \frac{1}{x_n}$, we get $x = 2 - \frac{1}{x}$, or equivalently,

$$0 = x - 2 + \frac{1}{x} = \frac{x^2 - 2x + 1}{x} = \frac{(x-1)^2}{x}.$$

This shows that $x = 1$, i.e. $\lim x_n = 1$.

HW 8.3. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers given as follows

$$a_1 \leq a_2 \leq \cdots \leq a_n \cdots b_n \leq \cdots b_2 \leq b_1,$$

Show that both $\lim a_n$ and $\lim b_n$ exist, and $\lim a_n \leq \lim b_n$.

Proof: Since $a_1 \leq a_2 \leq \cdots \leq a_n \cdots b_n \leq \cdots b_2 \leq b_1$, the sequence $\{a_n\}$ is monotone increasing and bounded above by b_k for each $k \in \mathbb{N}$. By Monotone Convergence theorem, $\lim a_n$ exists, and satisfies

$$\lim a_n = \sup\{a_n\} \leq b_k$$

for all $k \in \mathbb{N}$. This shows that $\lim a_n$ is a lower bound of $\{b_k\}$. Since the sequence $\{b_k\}$ is decreasing and bounded below by $\lim a_n$, we get

$$\lim a_n = \sup\{a_n\} \leq \inf\{b_k\} = \lim b_k.$$

HW 8.4. Let $s_n = \frac{1}{1^2} + \frac{1}{2^2} \cdots + \frac{1}{n^2}$ for each $n \in \mathbb{N}$. Show that the sequence $\{s_n\}$ is increasing and bounded above. Is $\{s_n\}$ a convergent sequence ? Explain.

Proof: It is clear that $\{s_n\}$ is monotone increasing since $s_{n+1} - s_n = \frac{1}{(n+1)^2} > 0$.

Now the sequence $\{s_n\}$ is bounded above since for any $n \geq 2$,

$$\begin{aligned} s_n &= \frac{1}{1^2} + \frac{1}{2^2} \cdots + \frac{1}{n^2} \\ &< 1 + \frac{1}{1 \cdot 2} + \cdots + \frac{1}{(n-1) \cdot n} \\ &= 1 + (\frac{1}{1} - \frac{1}{2}) + \cdots + (\frac{1}{(n-1)} - \frac{1}{n^2}) = 2 - \frac{1}{n^2} < 2. \end{aligned}$$

By Montone Convergence theorem, $\{s_n\}$ is a convergent sequence, i.e. $\lim_{n\to\infty} s_n$ exists. We can express

$$\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

HW 8.5. Show that the sequence $\{x_n\}$ with $x_n = 1 - (-1)^n + \frac{1}{n}$ is divergent.

Proof: The even terms form a subsequence $x_{2k} = 1 - (-1)^{2k} + \frac{1}{2k} = \frac{1}{2k} \to 0$.

The odd terms form a subsequence $x_{2k-1} = 1 - (-1)^{2k-1} + \frac{1}{2k-1} = 2 + \frac{1}{2k-1} \to 2$. Since these two subsequences converge to different numbers, $\{x_n\}$ is a divergent sequence.

HW 8.6. Show that the sequence $\{x_n\}$ with $x_n = \sin(\frac{n\pi}{4})$ is divergent.

Proof: This sequence has two subsequences $x_{8k} = \sin(\frac{8k\pi}{4}) = \sin(2k\pi) = 0 \to 0$,

and $x_{8k+2} = \sin(\frac{(8k+2)\pi}{4}) = \sin(2k\pi + \frac{\pi}{2}) = 1 \to 1$.

Therefore, the sequence is divergent.