Math347, Spring 2018, Homework #7, Solution
Due to Wednesday, February 14, 2018

HW7.1. Use the definition to prove that $\lim_{n\to\infty}\sqrt{1+\frac{1}{n}}=1$.

Discussion: We want to show:

For every $\varepsilon>0$, there exists a positive integer $K\in\mathbb{N}$ such that for all $n\geq K$

$$\left|\sqrt{1+\frac{1}{n}}-1\right|=\frac{\left|(\sqrt{1+\frac{1}{n}}-1)(\sqrt{1+\frac{1}{n}}+1)\right|}{\sqrt{1+\frac{1}{n}}+1}=\frac{\frac{1}{n}}{\sqrt{1+\frac{1}{n}}+1}<\frac{1}{n}<\varepsilon.$$

So we can take $K>\frac{1}{\varepsilon}$.

Proof: For every $\varepsilon>0$, there exists a positive integer $K>\frac{1}{\varepsilon}$ such that for all $n\geq K$

$$\left|\sqrt{1+\frac{1}{n}}-1\right|=\frac{\left|(\sqrt{1+\frac{1}{n}}-1)(\sqrt{1+\frac{1}{n}}+1)\right|}{\sqrt{1+\frac{1}{n}}+1}=\frac{\frac{1}{n}}{\sqrt{1+\frac{1}{n}}+1}<\frac{1}{n}\leq\frac{1}{K}<\varepsilon.$$

This shows that $\lim_{n\to\infty}\sqrt{1+\frac{1}{n}}=1$.

HW7.2. Use the definition to show $\lim_{n\to\infty}\left(\frac{n^2-1}{2n^2+3}\right)=\frac{1}{2}$.

Discussion: We want to show:

For every $\varepsilon>0$, there exists a $K\in\mathbb{N}$ such that for all $n\geq K$

$$\left|\frac{n^2-1}{2n^2+3}-\frac{1}{2}\right|=\left|\frac{2(n^2-1)-(2n^2+3)}{2(2n^2+3)}\right|=\frac{5}{4n^2+6}<\frac{8}{4n^2}<\frac{2}{n}<\varepsilon.$$

So we can take $K>\frac{2}{\varepsilon}$.

Proof: For every $\varepsilon>0$, there exists a positive integer $K>\frac{2}{\varepsilon}$

$$\left|\frac{n^2-1}{2n^2+3}-\frac{1}{2}\right|=\left|\frac{2(n^2-1)-(2n^2+3)}{2(2n^2+3)}\right|=\frac{5}{4n^2+6}<\frac{8}{4n^2}<\frac{2}{n}\leq\frac{2}{k}<\varepsilon.$$

HW7.3. Use definition to show that $\lim_{n\to\infty}x_n=0$ if and only if $\lim_{n\to\infty}|x_n|=0$.

Proof: Suppose that $\lim_{n\to\infty}x_n=0$. Then for every $\varepsilon>0$, there exists a $K\in\mathbb{N}$ such that for all $n\geq K$ we have $|x_n-0|<\varepsilon$. This implies that

$$||x_n|-0|=|x_n|=|x_n-0|<\varepsilon.$$

This shows that $\lim_{n\to\infty}|x_n|=0$.

On the other hand, suppose that $\lim_{n\to\infty}|x_n|=0$. Then for every $\varepsilon>0$, there exists a $K\in\mathbb{N}$ such that for all $n\geq K$ we have

$$|x_n-0|=||x_n|-0|<\varepsilon.$$

This shows that $\lim_{n\to\infty}x_n=0$.

HW7.4. Prove that if $\lim x_n = x > 0$, then there exists a natural number M such that $x_n > 0$ for all $n \geq M$.

Proof : Let $\varepsilon = x > 0$. Since $\lim x_n = x$, there exists a positive integer $M = K$ such that for all $n \geq M$, $|x_n - x| < \varepsilon$. In this case, we have

$$0 = x - \varepsilon < x_n < x + \varepsilon.$$

HW7.5. If (b_n) is a bounded sequence and $\lim a_n = 0$. Show that $\lim a_n b_n = 0$.

Proof: Since (b_n) is bounded, there exists a positive number $M > 0$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$.

Now since $\lim a_n = 0$, for $\forall \varepsilon > 0$, there exists a positive integer $K \in \mathbb{N}$ such that for all $n \geq K$, we have

$$|a_n| = |a_n - 0| < \frac{\varepsilon}{M}.$$

It follows that for all $n \geq K$, we have

$$|a_n b_n - 0| = |a_n b_n| = |a_n||b_n| < \frac{\varepsilon}{M} M = \varepsilon.$$

This shows that $\lim a_n b_n = 0$.

HW7.6. Let $\{x_n\}$ be a sequence of real numbers.

a) State the definition that the sequence $\{x_n\}$ does not converge to x.
Definition: The sequence $\{x_n\}$ does not converge to x if there exists a $\varepsilon_0 > 0$ such that for any positive integer k, there exists a positive integer $n_k \geq k$ such that

$$|x_{n_k} - x| \geq \varepsilon_0.$$

b) Use definition to show that the sequence $x_n = (-1)^n + \frac{1}{n}$ does not converge to $x = 1$.

Observation: Look at odd terms $x_{2k-1} = -1 + \frac{1}{2k-1} \leq 0$. Since the distance of these terms from number 1 is greater than or equal to $x = 1$. So the sequence (x_n) can not converge to 1, and this also suggests that we can take $\varepsilon_0 = 1$ for our proof.

Proof: There exists $\varepsilon_0 = 1 > 0$ such that for all $k \in \mathbb{N}$, there exists $n_k = 2k - 1 \geq k$ such that

$$|x_{2k-1} - 1| = \left| \left(-1 + \frac{1}{2k-1} \right) - 1 \right| = 2 - \frac{1}{2k-1} \geq 1 = \varepsilon_0.$$

This shows that $x_n = (-1)^n + \frac{1}{n}$ does not converge to 1.