Math347, Spring 2018, Homework #6 Solution
Due to Wednesday, March 7, 2018

HW 6.1. Let $S = \{2 + \frac{1}{n^2} : n \in \mathbb{N}\}$ be a bounded sequence of real numbers. Use definition to verify that $\inf(S) = 2$ and $\sup(S) = 3$.

Verify $\inf(S) = 2$**:** Since $2 < 2 + \frac{1}{n^2} \in S$ for all $n \in \mathbb{N}$, 2 is a lower bound of S. Now let us consider condition (2'). For any $v > 2$, we get $\varepsilon = \sqrt{v-2} > 0$, and thus by the corollary of Archimedean Property, there exists a positive integer $n \in \mathbb{N}$ such that

$$0 < \frac{1}{n} < \varepsilon = \sqrt{v-2} \text{ or equivalently } \frac{1}{n^2} < \varepsilon^2 = v - 2.$$

This implies that there exists $n \in \mathbb{N}$ such that $2+\frac{1}{n^2} < v$. Since $2+\frac{1}{n}$ is an element in S, v is not a lower bound of S. This shows that $2 = \inf(S)$.

Verify $\sup(S) = 3$**:** This part is very easy since 3 is the maximum (the largest element) of S. So 3 is an upper bound of S, and if M' is an upper bound of S. then we have $3 \le M'$. This shows that $3 = \sup(S)$.

HW 6.2. If $y > 0$, show that there exists $n \in \mathbb{N}$ such that $\frac{1}{2^n} < y$.

Proof: By the Archimedean property, there exists $n \in \mathbb{N}$ such that $-\log_2 y < n$. This implies that $\frac{1}{y} < 2^n$ and thus $\frac{1}{2^n} < y$.

Another Proof: By the corollary of Archimedean property, for $y > 0$, there exists $n \in \mathbb{N}$ such that $0 < 1/n < y$. Since $2^n = (1 + 1)^n > n$ (or prove this by Indiction), we get

$$0 < 1/2^n < 1/n < y.$$

HW 6.3. Let $u > 0$ be a positive real number. Show that for any real numbers $x < y$, there exists a rational number r such that

$$x < ru < y.$$

Proof: Since $u > 0$ and $x < y$, we get $x/u < y/u$. By the density of rational numbers, there exists a rational number r such that $x/u < r < y/u$. Therefore, we have $x < ru < y$.

HW 6.4. Let $S \subseteq \mathbb{R}$ be a bounded subset of $\mathbb{R}$, and let $I_S = [\inf(S), \sup(S)]$.
1) Show that $S \subseteq I_S$.

Proof: Since

$$\inf(S) \le s \le \sup(S)$$

for every $s \in S$, we have $S \subseteq [\inf(S), \sup(S)]$.

2) If J is any closed bounded interval containing S, show that J contains I_S.

Proof: Let $J = [a, b]$. Since $S \subseteq J$, then we have

$$a \le s \le b$$

for every $s \in S$. This shows that a is a lower bounded of S and b is an upper bound of S. By definition, we have $a \leq \inf(S)$ and $\sup(S) \leq b$. Therefore, we must have $I_S \subseteq J$.

HW 6.5. Prove that $\cap_{n=1}^{\infty}[0, \frac{1}{n}] = \{0\}$.

Proof: Since $0 \in [0, \frac{1}{n}]$ for all $n \in \mathbb{N}$, it is clear that $\{0\} \subseteq \cap_{n=1}^{\infty}[0, \frac{1}{n}]$.

On the other hand, let x be an element in $\cap_{n=1}^{\infty}[0, \frac{1}{n}]$. We want to show $x = 0$. Suppose not. Then we must have $x > 0$. By the corollary of Archimedean property, there exists a positive integer n_0 such that $\frac{1}{n_0} < x$. Then x does not belong to $[0, \frac{1}{n_0}]$ and thus x does not belong to $\cap_{n=1}^{\infty}[0, \frac{1}{n}]$, a contradiction. So we must have $\cap_{n=1}^{\infty}[0, \frac{1}{n}] = \{0\}$.

HW 6.6. Let $S = \{x : \text{irrational in } [0, 1]\}$ be the set of irrational numbers in $[0, 1]$. Use definition to verify that $\sup(S) = 1$.

Proof: 1) It is clear that 1 is an upper bound of S.

2') For any $u < 1$ (we can consider $0 < u < 1$), it is known from the density of irrational numbers that there exists an irrational number x such that $u < x < 1$. Then x is an element in S and this shows that u is not an upper bound of S. Therefore, we must have $\sup(S) = 1$.