Math347, Spring 2018, Homework #5, Solution
Due to Wednesday, February 28, 2018

HW 5.1. Determine whether S is bounded and determine $\sup(S)$ and $\inf(S)$, if they exist.

a) $S = \{x : x^2 < 5x\}$
Solution: We find $S = (0, 5)$. Then $\inf(S) = 0$ and $\sup(S) = 5$.
We can use definition to verify this, using conditions (1) and (2').

b) $S = \{x : 2x^2 < x^3 + x\}$.
Solution: We find $S = \{x : x > 0 \text{ and } x \neq 1\} = (0, 1) \cup (1, \infty)$.
We can use definition to verify that S is bounded below with $\inf(S) = 0$.
But it it is not bounded above and thus S does not have the least upper bound.

HW 5.2. If a set $S \subseteq \mathbb{R}$ contains one of its upper bound, show this upper bound is the supremum of S.

Proof: Assume that u is an upper bound of S and u is contained in S. Then

1) u is an upper bound for S (this is given).
2) if M' is an upper bound of S, we have $u \leq M'$ since u is an element in S.

By definition, this shows that u is the least upper bound of S.

HW 5.3. Let $S \subseteq \mathbb{R}$ be nonempty. Show that if $u = \sup(S)$, then for every number $n \in \mathbb{N}$ the number $u - \frac{1}{n}$ is not an upper bound of S, but the number $u + \frac{1}{n}$ is an upper bound of S.
Proof: First let us show: $u - \frac{1}{n}$ is not an upper bound of S. Since $u - \frac{1}{n} < u$, if we assume that $u - \frac{1}{n}$ is an upper bound of S, then u is NOT the least upper bound of S, a contradiction. This shows that $u - \frac{1}{n}$ is not an upper bound of S (for any $n \in \mathbb{N}$).

Next, we show: $n + \frac{1}{n}$ is an upper bound of S. Since u is an upper bound of S, we have $s \leq u < u + \frac{1}{n}$ for all $s \in S$. This shows that $u + \frac{1}{n}$ is an upper bound of S.

HW 5.4. Let S be a bounded set in $\mathbb{R}$ and let S_0 be a nonempty subset of S. Show that

$$\inf(S) \leq \inf(S_0) \leq \sup(S_0) \leq \sup(S).$$

Proof: Let $\emptyset \neq S_0 \subseteq S$ be bounded subsets of $\mathbb{R}$. Then for any $s_0 \in S_0$, we have $s_0 \in S$ and thus $s_0 \leq \sup(S)$. This shows that $\sup(S)$ is an upper bound of S_0. Since $\sup(S_0)$ is the least upper bound of S_0, we must have

$$\sup(S_0) \leq \sup(S).$$

Similarly, for any $s_0 \in S_0$, we have $s_0 \in S$ and thus $s_0 \geq \inf(S)$. This shows that $\inf(S)$ is a lower bound of S_0. Since $\inf(S_0)$ is the greatest lower bound of S_0, we have

$$\inf(S) \leq \inf(S_0).$$

Since $\inf(S_0) \leq \sup(S_0)$, we get

$$\inf(S) \leq \inf(S_0) \leq \sup(S_0) \leq \sup(S).$$

HW 5.5 If $S = \{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\}$, find $\inf(S)$ and $\sup(S)$.

Solution: Consider $n = 1, 2, \cdots$, we get sequences $\{1 - \frac{1}{m}\}$, $\{\frac{1}{2} - \frac{1}{m}\}$, $\cdots$. We see that 1 is an upper bound, and we claim that $\sup(S) = 1$. We only need to verify condition (2') given in definition: For any $u < 1$, we have $\epsilon = 1 - u > 0$. It is known from the Archimedean property that there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < 1 - u$, or equivalently, $u < 1 - \frac{1}{m}$. Therefore, any $u < 1$ is not an upper bound of S. This shows that $\sup(S) = 1$.

Considering $m = 1, 2, \cdots$, we get sequences $\{\frac{1}{n} - 1\}$, $\{\frac{1}{n} - \frac{1}{2}\}$, $\cdots$. It is clear that -1 is a lower bound of S. For any $-1 < v$, we get $v + 1 > 0$ and thus there exists a positive integer $n \in \mathbb{N}$ such that $\frac{1}{n} < v + 1$. It follows that we get $\frac{1}{n} - 1 \in S$ such that $\frac{1}{n} - 1 < v$. This shows that $\inf(S) = -1$.

HW 5.6. Let (a_n) and (b_n) be two bounded seuqences. Show that

$$\sup(a_n + b_n) \leq \sup(a_n) + \sup(b_n).$$

Proof: For each $n \in \mathbb{N}$, we have

$$a_n \leq \sup(a_n) \text{ and } b_n \leq \sup(b_n).$$

Then $a_n + b_n \leq \sup(a_n) + \sup(b_n)$. This shows that $\sup(a_n) + \sup(b_n)$ is an upper bound of $(a_n + b_n)$. Since $\sup(a_n + b_n)$ is the the least upper boundand of $(a_n + b_n)$, we get

$$\sup(a_n + b_n) \leq \sup(a_n) + \sup(b_n).$$

Remark: In this case, we can find two bounded sequences $a_n = (-1)^{n-1}$ and $b_n = (-1)^n$ such that

$$\sup(a_n + b_n) = 0 < \sup(a_n) + \sup(b_n) = 2.$$