Introduction

While much is known about solutions of the equation $\Delta u = 0$ on a domain $D$, very little seems to be known about solutions of the equation $(-\Delta)^\alpha u = 0$ for $0 < \alpha < 1$. Since $(-\Delta)^\alpha$ is just a fractional power of the Laplacian, one might expect the analysis of these two equations to be quite similar, but standard techniques yielding insight into the first equation often yield little or no information about the second. In fact, this is because $(-\Delta)^\alpha$ is no longer a differential operator, but is instead an integro-differential operator. We approach $(-\Delta)^\alpha$ as the infinitesimal generator of a Markov process on a domain $D$ obtained by subordinating Brownian motion. From the probabilistic point of view, new difficulties are introduced because the subordinated process does not have continuous trajectories like Brownian motion. Based on this characterization and the associated potential theory, we characterize the solutions of $(-\Delta)^\alpha u = 0$ as a certain class of "harmonic" functions, and we give conditions under which the solution is continuous.

In order to make these results accessible to both probabilists and analysts, we discuss the necessary potential theory in section 1. Probabilists...
will recognize the facts in this section as statements about subordinations couched in analytic terms. We apply these facts in section 2 to establish an integral representation which provides an explicit one-to-one correspondence between the $\Delta$-harmonic functions associated with Brownian motion on $D$ and the $(-\Delta)^{\alpha}$-harmonic functions associated with the subordinated process on $D$. In addition, continuity of these harmonic functions is established.

1 Definitions and General Results

Let $D$ be a domain in $\mathbb{R}^n$, i.e., $D$ is an open and connected subset of $\mathbb{R}^n$. Let $(X_t; t \geq 0)$ be a Brownian motion killed upon exit from $D$. By $\tau = \tau_D$ we denote the first exit time from $D$.

For an additive functional $A$ we define its $\Gamma_\alpha$-potential by

$$\Gamma_{\alpha,A}(x) = \frac{1}{\Gamma(\alpha)} E^x \left[ \int_0^\infty t^{\alpha-1} dA_t \right], \quad x \in D, 0 < \alpha < \infty. \quad (1)$$

An easy application of the Markov property and Fubini's theorem shows that the following formula is valid: for any $\beta > 0$,

$$\frac{1}{\Gamma(\beta)} E^x \left[ \int_0^\infty s^{\beta-1} \Gamma_{\alpha,A}(X_s) ds \right] = \Gamma_{\alpha+\beta,A}(x). \quad (2)$$

The next proposition, which is a consequence of (2), will be used in the sequel several times.

**Proposition 1** Suppose $\Gamma_{\alpha,A}$ is not identically infinite. Then $\Gamma_{\beta,A}$ is locally integrable for all $0 < \beta \leq \alpha$.

**Proof.** If $\beta < \alpha$, then, by (2), $\Gamma_{\beta,A}$ is also not identically infinite. Therefore it is enough to prove the statement for $\beta = \alpha$. Notice that in the case $\alpha = 1$, we have an ordinary potential, thus the statement of the proposition is correct.

Consider the case $0 < \alpha < 1$. We claim that, for every $t > 0$, the following inequality is valid:

$$\alpha \int_0^t e^{-s}(t-s)^{\alpha-1} ds \leq t^{\alpha-1}. \quad (3)$$

For $t \leq 1$, the left hand side is smaller than

$$\alpha \int_0^t (t-s)^{\alpha-1} ds = t^\alpha \leq t^{\alpha-1}. $$
To prove (3) for $t \geq 1$ now, it is enough to prove that the following function

$$t \mapsto e^{t^{\alpha-1}} - \alpha \int_0^t e^{s^{\alpha-1}} ds$$

is increasing on $[1, \infty)$. This follows immediately by differentiation.

Consider now the resolvent operator $G_1$ for the process $(X_t)$. Using (3) we obtain

$$(G_1 \Gamma_{\alpha,A})(x) = E^x \left[ \int_0^\infty e^{-t} \Gamma_{\alpha,A}(X_t) dt \right]$$

$$= \frac{1}{\Gamma(\alpha)} E^x \left[ \int_0^\infty e^{-t} \int_t^\infty (s-t)^{\alpha-1} dA_s dt \right]$$

$$= \frac{1}{\Gamma(\alpha)} E^x \left[ \int_0^\infty dA_s \int_0^s e^{-t}(s-t)^{\alpha-1} dt \right]$$

$$\leq \frac{1}{\Gamma(\alpha)} \frac{1}{\alpha} E^x \left[ \int_0^\infty s^{\alpha-1} dA_s \right] = \frac{1}{\alpha} \Gamma_{\alpha,A}(x),$$

where in the first line above we used the Markov property, in the second line we used Fubini's theorem and in the third line we used (3). Since $(X_t)$ is the Brownian motion killed upon exit from $D$, $G_1$ has a density. Therefore, a standard proof (see, for example, page 267 of [1]) shows that $\Gamma_{\alpha,A}$ is locally integrable, since it is not identically infinite.

It remains to prove the statement for $\alpha > 1$. Applying (2) to $1$ and $\alpha = 1 > 0$, we obtain

$$\Gamma_{\alpha,A}(x) = E^x \left[ \int_0^\infty \Gamma_{\alpha-1,A}(X_s) ds \right].$$

Hence $\Gamma_{\alpha,A}$ is an ordinary potential, and, since it is not identically infinite, it must be locally integrable.

$q.e.d.$

The next important property that we expect for $\Gamma$-potentials is that they uniquely determine additive functionals. In proving the uniqueness theorem for $\Gamma$-potentials we will use the following, easily derived, formula:

$$\Gamma_{\alpha,A}(x) = \begin{cases} 
\frac{1-\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-2} E^x[A_t] dt; & 0 < \alpha < 1 \\
E^x[A_\infty]; & \alpha = 1 \\
\frac{\alpha-1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-2} E^x[A_\infty - A_t] dt; & \alpha > 1.
\end{cases} \quad (4)$$

**Theorem 1** If, for some $\alpha \in (0, \infty)$, $\Gamma_{\alpha,A}$ and $\Gamma_{\alpha,B}$ are not identically infinite, and $\Gamma_{\alpha,A} = \Gamma_{\alpha,B}$, then $A = B$. 
Proof. Notice that, by Proposition 1, $\Gamma_{\alpha,A}$ and $\Gamma_{\alpha,B}$ are finite almost everywhere. Hence, if $\alpha = 1$, then $A = B$, by (4) and the uniqueness theorem for ordinary potentials ([1], Chapter IV, Theorem 2.13).

Consider the case $0 < \alpha < 1$. It follows, by (4), that for almost every $x$ and for almost every $t$, $E^x[A_t]$ and $E^x[B_t]$ are finite. Since they are nondecreasing functions in $t$, we conclude that for almost every $x$, $E^x[A_t]$ and $E^x[B_t]$ are finite for every $t$. Therefore, for almost every $x$, $t \mapsto E^x[A_t]$ and $t \mapsto E^x[B_t]$ determine $\sigma$-finite measures on $[0, \infty)$. Since $\Gamma_{\alpha,A} = \Gamma_{\alpha,B}$, it follows that $E^x[\Gamma_{\alpha,A}(X_t)] = E^x[\Gamma_{\alpha,B}(X_t)]$. Thus by Fubini's theorem we have

$$E^x[\int_t^\infty (s - t)^{\alpha - 1}dA_s] = E^x[\int_t^\infty (s - t)^{\alpha - 1}dB_s].$$

Since $s \mapsto (s - t)^{\alpha - 1}$ is deterministic, we obtain that for almost every $x$, and, for every $t$,

$$\int_t^\infty (s - t)^{\alpha - 1}dE^x[A_s] = \int_t^\infty (s - t)^{\alpha - 1}dE^x[B_s].$$

Recall (see, for example, [4], page 3.31) that $s \mapsto (s - t)^{\alpha - 1}$, $0 < \alpha < 1$, is a potential kernel on $[0, \infty)$ and determines measures uniquely. Therefore, for almost every $x$,

$$E^x[A_t] = E^x[B_t] < \infty, \text{ for every } t.$$

It follows (see [1], page 159) that $A = B$.

Assume now that the statement is proved for every $0 < \alpha \leq n$, where $n$ is a positive integer. We will show that then it must be valid for every $n < \alpha \leq n + 1$. Since $\alpha > 1$, it follows, by (4) and Proposition 1, that for almost every $x$, $E^x[A_\infty]$ and $E^x[B_\infty]$ are finite. Since $\Gamma_{\alpha,A} = \Gamma_{\alpha,B}$, it follows, by (2),

$$E^x[\int_0^\infty s^{(\alpha - 1) - 1}\Gamma_{1,A}(X_s)ds] = E^x[\int_0^\infty s^{(\alpha - 1) - 1}\Gamma_{1,B}(X_s)ds],$$

and both sides are finite for almost every $x$. Notice that these are $\Gamma_{\alpha-1}$ potentials of $\int_0^t \Gamma_{1,A}(X_s)ds$ and $\int_0^t \Gamma_{1,B}(X_s)ds$, respectively. Since $\alpha - 1 \leq n$, it follows that the two additive functionals are equivalent, by the inductive assumption. Since $\Gamma_{1,A}(x) = E^x[A_\infty]$, it follows that, for almost every $x$,

$$E^x[A_\infty] = E^x[B_\infty] < \infty,$$

which proves the theorem.
Notice that one of the consequences of the proof of Proposition 1 (case $\alpha > 1$) is that for $\alpha > 1$,

$$u(x) = \Gamma_{\alpha,A}(x)$$

is superharmonic. For this superharmonic function the following is true:

$$\int_0^s t^{\alpha-1} dA_t + u(X_s) \leq E^x[\int_0^\infty t^{\alpha-1} dA_t | \mathcal{F}_s].$$

From (6) we can get that for any $\lambda > 0$,

$$\lambda P^x[u^* > \lambda] \leq \lambda P^x[M^* > \lambda] \leq E^x[\int_0^\infty t^{\alpha-1} dA_t],$$

where $M$ is the martingale

$$M_s = E^x[\int_0^\infty t^{\alpha-1} dA_t | \mathcal{F}_s],$$

and $^*$ is the usual notation for the supremum.

**Remark 1.** The definition of the $\Gamma$-potential and the results of this section can be obtained for much more general Markov processes $(X_t)$. The assumptions needed are that the resolvent of $(X_t)$ has a density with the usual properties (see, for example, [1], Chapter VI) and the additive functionals are natural (in the sense of [1], Chapter IV).

However, the main results of this paper (contained in the following sections) are obtained only for Brownian motion killed upon exit from $D$. For this reason we did not introduce $\Gamma$-potentials in the most general setting.
2 Representations of Harmonic Functions

It is possible to represent positive harmonic functions as \( \Gamma_\alpha \)-potentials for \( 0 < \alpha < 1 \). Indeed, let \( s = U f \), where \( U \) is the potential kernel of Brownian motion killed upon exit from \( D \) and \( f \geq 0 \). Let us denote the \( \Gamma_\alpha \)-potential, with respect to the additive functional 

\[
A_t = \int_0^t f(X_s) \, ds,
\]

by \( V^\alpha f \), i.e.,

\[
V^\alpha f(x) = \frac{1}{\Gamma(\alpha)} E^x \left[ \int_0^\infty t^{\alpha-1} f(X_t) \, dt \right]. \tag{7}
\]

By (2), we get, for every \( 0 < \alpha < 1 \),

\[
V^\alpha V^{1-\alpha} f(x) = U f(x). \tag{8}
\]

As usual, we use \( P_t \) to denote the transition semigroup of \( (X_t) \). Then we obtain

\[
\Gamma(1 - \alpha) V^{1-\alpha} f(x) = E^x \int_0^\infty t^{-\alpha} f(X_t) \, dt
= \int_0^\infty t^{-\alpha} P_t f(x) \, dt
= \int_0^\infty P_t f(x) \, dt \int_0^\infty \frac{1}{\Gamma(\alpha)} \lambda^{-(1-\alpha)} e^{-\lambda t} d\lambda
= \frac{1}{\Gamma(\alpha)} \int_0^\infty \lambda^{\alpha-1} d\lambda \int_0^\infty e^{-\lambda t} P_t f(x) \, dt
= \frac{1}{\Gamma(\alpha)} \int_0^\infty \lambda^{\alpha-1} s^\lambda \, d\lambda,
\]

where \( s^\lambda = U^\lambda f \) satisfies

\[
s = s^\lambda + \lambda U^\lambda s. \tag{9}
\]

Thus we can rewrite (8) as

\[
U f = V^\alpha g, \quad g = \frac{1}{\Gamma(\alpha)\Gamma(1 - \alpha)} \int_0^\infty \lambda^{\alpha-1} s^\lambda \, d\lambda. \tag{10}
\]

Now we are going to extend (10) to more general excessive functions. For any excessive function \( s \), it is easy to show (see, for example, [5]) that for each \( \lambda > 0 \), there is a unique \( \lambda \)-excessive function \( s^\lambda \) such that

\[
s = s^\lambda + \lambda U^\lambda s. \tag{11}
\]
And if \( s_n = U f_n \) increases to \( s \), \( s_n^\lambda = U^\lambda f_n \) tends to \( s^\lambda \) almost everywhere. Applying Fatou's lemma and (10) to this fact, we obtain that, for every \( 0 < \alpha < 1 \), and, for any excessive function \( s \),

\[
V^\alpha \left[ \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \lambda^{\alpha-1} s^\lambda d\lambda \right] \leq s. \tag{12}
\]

We can show that for a special class of excessive functions (12) becomes an equality.

An excessive function \( s \) is called purely excessive if

\[
\lim_{t \to \infty} P_t s = 0,
\]

almost everywhere. If \( s \) is purely excessive, (11) can be expanded to

\[
s = s^\lambda + \lambda U^\lambda s = s^\lambda + \lambda U s^\lambda, \tag{13}
\]

and for any \( \lambda, \mu \geq 0 \),

\[
U^\lambda s^\mu = U^\mu s^\lambda. \tag{14}
\]

From (14) we know that \( U^1 s \) is the potential of \( s^1 \), i.e., \( U^1 s = U s^1 \). Therefore

\[
(U^1 s)^\lambda = U^\lambda s^1 = U^1 s^\lambda. \tag{15}
\]

For potentials of functions, (10) is true. Thus from (15)

\[
U^1 s = U s^1 = V^\alpha \left[ \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \lambda^{\alpha-1} U^1 s^\lambda d\lambda \right] = U^1 \left[ V^\alpha \left( \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \lambda^{\alpha-1} s^\lambda d\lambda \right) \right].
\]

Applications of (12) now shows that we just proved the following result.

**Proposition 2** If \( s \) is purely excessive, then, for every \( 0 < \alpha < 1 \),

\[
s = V^\alpha g, \quad g = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \lambda^{\alpha-1} s^\lambda d\lambda. \tag{16}
\]

**Remark 2.** If \( D \) is a bounded \( C^2 \)-domain in \( R^n \), every positive harmonic function is purely excessive. Hence, Proposition 2 represents every positive harmonic function on a nice domain as \( \Gamma_\alpha \)-potentials of functions.
It is possible to represent $s$ in (16) in terms of $s$ directly. Suppose that $s$ is purely excessive. By (13), we get

$$s^\lambda = \lambda \int_0^\infty (s - P_t s) e^{-\lambda t} dt.$$ 

Thus

$$\int_0^\infty \lambda^{\alpha-1} s^\lambda d\lambda = \lambda^{\alpha} \int_0^\infty (s - P_t s) e^{-\lambda t} dt$$

$$= \lambda^{\alpha} \int_0^\infty (s - P_t s) dt \int_0^\infty \lambda^{\alpha} e^{-\lambda t} d\lambda$$

$$= \Gamma(\alpha + 1) \int_0^\infty t^{-\alpha} (s - P_t s) \frac{dt}{t}.$$ 

Hence, if $s$ is purely excessive, then $s = V^\alpha g$, where

$$g = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty t^{-\alpha} (s - P_t s) \frac{dt}{t}.$$  

(17)

Let us now apply (16) and (17) to some special excessive functions and compute $g$ explicitly.

**Example 1.** Let $D$ be a bounded domain. Then $s \equiv 1$ is purely excessive, and we can apply (16). Since

$$s^\lambda(x) = E^x[e^{-\lambda \tau}]$$

(recall that $\tau$ is the first exit time from $D$), we get

$$g(x) = \frac{1}{\Gamma(1 - \alpha)} E^x[\tau^{-\alpha}].$$  

(18)

Also, since $1 = V^\alpha g$, we get

$$\int_D dx = \int_D V^\alpha g(x) dx = \int_D g(x) V^\alpha 1(x) dx$$

$$= \int_D g(x) \frac{1}{\alpha \Gamma(\alpha)} E^x[\tau^\alpha] dx.$$ 

Thus for each $0 < \alpha < 1$,

$$\int_D E^x[\tau^{-\alpha}] E^x[\tau^\alpha] dx = \alpha \Gamma(\alpha) \Gamma(1 - \alpha) \int_D dx.$$  

(19)
Example 2. The assumptions are the same as in the previous example. Let us now compute \( g \) for \( s(x) = E^x[\tau^m] \), where \( m \) is a positive integer.

We will use the following formula, which can be easily proved by induction: for every positive integer \( m \), and for every real non-integer \( a \),

\[
\sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^{m-k}}{a-(m-k)} = \frac{-m!}{\prod_{k=0}^{m}(k-a)} = \frac{(-1)^m}{a \cdot \binom{m}{a-1}}. \tag{20}
\]

By the Markov property, we get

\[
P_t s(x) = E^x[E^X(t)(\tau^m); t < \tau] = E^x[\tau^m \circ \theta_t; t < \tau] = E^x[(\tau - t)^m; t < \tau] = \sum_{k=0}^{m} \binom{m}{k} (-t)^{m-k} E^x[\tau^k; t < \tau].
\]

Therefore

\[
\frac{(s - Pt s)(x)}{t} = \frac{E^x[\tau^m; \tau \leq t]}{t} - \sum_{k=0}^{m-1} \binom{m}{k} (-1)^{m-k} t^{m-k-1} E^x[\tau^k; t < \tau].
\]

Applying Fubini’s theorem we get

\[
\int_0^\infty t^{-\alpha} \frac{E^x[\tau^m; \tau \leq t]}{t} dt = E^x[\tau^m] \int_0^\infty t^{-\alpha-1} \mathbf{1}_{(t \leq \tau)} dt = \frac{E^x[\tau^{m-\alpha}]}{\alpha},
\]

and, similarly, for \( k = 0, 1, \ldots, m - 1 \),

\[
\int_0^\infty t^{-\alpha} t^{m-k-1} E^x[\tau^k; t < \tau] dt = \frac{E^x[\tau^{m-\alpha}]}{(m-k) - \alpha}.
\]

Finally, using (17) and applying (20), we obtain

\[
g(x) = \frac{(-1)^m}{\Gamma(1 - \alpha) \binom{\alpha-1}{m}} E^x[\tau^{m-\alpha}]. \tag{21}
\]

Now let us restrict our attention to the case of a domain \( D \) such that all excessive functions on \( D \) are purely excessive (see Remark 2). The main result of this section is the following representation theorem:
Theorem 2 If $0 < \alpha < 1$ and if $s$ is $U$-excessive on $D$, then there is a $V^{1-\alpha}$-excessive function $g$ such that

$$s = V^\alpha g.$$  \hspace{1cm} (22)

Furthermore, if $s$ is $U$-harmonic, then $g$ is $V^{1-\alpha}$-harmonic.

Conversely, if $g$ is $V^{1-\alpha}$-excessive and $s = V^\alpha g$ is not identically infinite, then $s$ is $U$-excessive. If $g$ is $V^{1-\alpha}$-harmonic and $s$ is not identically infinite, then $s$ is $U$-harmonic.

Before we give the proof of this theorem, a lemma is needed.

**Lemma 1** Let $V$ be a potential kernel and suppose that $Vg_n \downarrow Vg$ where $g_n$ are $V$-excessive. Then $g$ is equal to an excessive function almost everywhere.

**Proof.** We have

$$Vg_n = V^\lambda g_n + \lambda V^\lambda Vg_n,$$  \hspace{1cm} (23)

where $V^\lambda$ is the resolvent of $V$. By the monotone convergence theorem we know that $\lim_{n \to \infty} V^\lambda Vg_n = V^\lambda Vg$. It follows, by (23), that

$$\lim_{n \to \infty} V^\lambda g_n(x) = V^\lambda g(x),$$  \hspace{1cm} (24)

if $Vg(x) < \infty$. Since $g_n$ is $V$-excessive, $\lambda V^\lambda g_n$ is increasing in $\lambda$. From (24) we conclude that if $Vg(x) < \infty$, then $\lambda V^\lambda g(x)$ is increasing in $\lambda$. Define

$$\tilde{g}(x) = \sup_{\lambda > 0} \lambda V^\lambda g(x).$$

Then, for all $x$ such that $Vg(x) < \infty$,

$$\tilde{g}(x) = \lim_{n \to \infty} nV^n g(x),$$

so, by monotone convergence, we have that for any $x$ such that $Vg(x) < \infty$,

$$\mu V^\mu \tilde{g}(x) = \lim_n \mu V^\mu nV^n g(x)$$
$$= \mu \lim_n nV^\mu V^n g(x)$$
$$= \mu \lim_n n[V^\mu g(x) - V^n g(x)]$$
$$= \mu V^\mu g(x) \leq \tilde{g}(x).$$
Thus, $\mu V^\mu \tilde{g}(x) \leq \tilde{g}(x)$ almost everywhere, and, by the excessivity of both sides, everywhere. Thus $\tilde{g}$ is supermedian and the first equality shows that $\tilde{g} = g$ almost everywhere. Hence $g$ is equal to an excessive function almost everywhere.

Q.E.D.

**Proof of the Theorem.** Notice that, by (8), if $g$ is $V^{1-\alpha}$-excessive, then $g$ is the monotone limit of $V^{1-\alpha} \phi_n$, and

$$V^\alpha g = \lim_{n \to \infty} V^\alpha V^{1-\alpha} \phi_n = \lim_{n \to \infty} U \phi_n,$$

i.e., $V^\alpha g$ is either identically infinite or $U$-excessive.

Let $s$ be $U$-excessive. Then $s$ is the increasing limit of potentials $U f_n$, i.e.,

$$s = \lim_n U f_n = \lim_n V^\alpha g_n$$

where $g_n = V^{1-\alpha} f_n$ are $V^{1-\alpha}$-excessive. From Proposition 2 we know that $s = V^\alpha g$, since $s$ is purely excessive. From Lemma 1 we conclude that $g$ is equal to a $V^{1-\alpha}$-excessive function almost everywhere.

Let $s$ be $U$-harmonic and $s = V^\alpha g$, where $g$ is $V^{1-\alpha}$-excessive. Then

$$g = V^{1-\alpha} \mu + \tilde{g},$$

where $\tilde{g}$ is $V^{1-\alpha}$-harmonic (and, of course, $V^{1-\alpha}$-excessive). We have, by (8),

$$s = V^\alpha g = V^\alpha V^{1-\alpha} \mu + V^\alpha \tilde{g} = U \mu + V^\alpha \tilde{g}.$$

Since $V^\alpha \tilde{g}$ is $U$-excessive and $s$ is $U$-harmonic, the potential part must be zero, i.e., $s = V^\alpha \tilde{g}$. It follows that $g = \tilde{g}$.

Finally, suppose that $g$ is $V^{1-\alpha}$-harmonic and $s = V^\alpha g$ is not identically infinite. Then $s$ is $U$-excessive and therefore

$$s = U \mu + h,$$

where $h$ is $U$-harmonic. By (8) and the proof above, there is a $V^{1-\alpha}$-harmonic function $\tilde{g}$ such that

$$V^\alpha g = s = V^\alpha (V^{1-\alpha} \mu + \tilde{g}).$$

It follows that $g = V^{1-\alpha} \mu + \tilde{g}$, and, since $g$ is $V^{1-\alpha}$-harmonic, that $g = \tilde{g}$, i.e., $s = h$.

Q.E.D.
As in the previous theorem, Let $D$ be a domain such that all excessive functions on $D$ are purely excessive. Assume that $h > 0$ is $U$-harmonic on $D$. We have proved that, for every $0 < \alpha < 1$, $h = V^\alpha g$, where

$$g = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty t^{-\alpha} \frac{(h - P_t h)}{t} \, dt$$

(25)

is $V^{1-\alpha}$-harmonic. Since $h$ is continuous, it is natural to ask if $g$ is also continuous. The following theorem answers this question.

**Theorem 3** Every positive $V^{1-\alpha}$-harmonic function $g$, such that $V^\alpha g$ is not identically infinite, is continuous.

**Proof.** Since Theorem 2 establishes an one-to-one correspondence between positive $U$-harmonic functions and positive $V^{1-\alpha}$-harmonic functions, it is enough to prove the statement of the theorem for $g$, which is of the form (25).

For every $\epsilon > 0$,

$$y \mapsto \int_0^\infty t^{-\alpha} \frac{(h - P_t h)(y)}{t} \, dt$$

is continuous, by the dominated convergence theorem, since $h$ and $P_t h$ are continuous (see [3]) and

$$\int_\epsilon^\infty t^{-\alpha - 1} \, dt = \frac{1}{\alpha \epsilon^\alpha} < \infty.$$

Hence, we need only to prove that

$$y \mapsto \int_0^\epsilon t^{-\alpha} \frac{(h - P_t h)(y)}{t} \, dt$$

is continuous at every $y \in D$. Let $x \in K \subset L^\circ$, where $K$ and $L$ are compact subsets of $D$ and as usual, $L^\circ$ is the interior of $L$. It is enough to prove that

$$\lim_{\epsilon \downarrow 0} \int_0^\epsilon t^{-\alpha} \frac{(h - P_t h)}{t} \, dt = 0$$

uniformly on $K$.

Let $\tau_L$ be the first exit time from $L$. Then, we have

$$(h - P_t h)(x) = h(x) - E^x[h(X_t); t < \tau_L] - E^x[h(X_t); \tau_L \leq t < \tau].$$
Recall that \((h(X_t)1_{\{t<\tau\}})\) is a positive martingale, since \(h\) is harmonic on \(D\). Since \(\tau_L\) is a stopping time, we get
\[
|((h - P_t h)(x))| \\
\leq |E^x[h(X_{\tau_L})] - E^x[h(X_{\tau_L}); t < \tau_L]| + |E^x[h(X_{\tau_L}); \tau_L \leq t < \tau]| \\
= |E^x[h(X_{\tau_L}); \tau_L \leq t]| + |E^x[h(X_{\tau_L}); \tau_L \leq t < \tau]|.
\]
Since \((X_t)\) is continuous and \(L\) is compact, it follows that \(X_{\tau_L} \in \partial L\) and that there exists a constant \(M = M(L, h) > 0\) such that \(h(X_{\tau_L}) \leq M\). Therefore, we get
\[
|(h - P_t h)(x)| \leq M[P^x(\tau_L \leq t) + P^x(\tau_L \leq t < \tau)] \\
\leq 2M P^x(\tau_L \leq t).
\]
Since \(K \subseteq L^0\) is compact, the distance \(d\) between \(K\) and \(\overline{L}\) is strictly positive. Thus, for every \(x \in K\), we have
\[
\{\tau_L \leq t\} \subseteq \{\sup_{0 \leq s \leq t} |X_s - x| \geq d\}
\]
almost surely with respect to \(P^x\). Using the fact that
\[
|X_s - x|^2 = \sum_{i=1}^{n} |X_s^i - x^i|^2
\]
is a \(P^x\)-submartingale, we obtain (by submartingale inequality)
\[
P^x(\tau_L \leq t) \leq P^x(\sup_{0 \leq s \leq t} |X_s - x|^2 \geq d^2) \leq \frac{E^x[|X_t - x|^2]}{d^2} \leq \frac{nt}{d^2},
\]
since \((X_t)\) is the Brownian motion killed upon exit from \(D\). This inequality completes the proof, since, for every \(x \in K\),
\[
|\int_0^t t^{-\alpha} \frac{(h - P_t h)(x)}{t} dt| \leq \int_0^t t^{-\alpha} \frac{2Mnt}{td^2} dt = \frac{2Mn}{(1 - \alpha)d^2} t^{1-\alpha}.
\]
\(Q.E.D.

3 Fractional Powers of the Laplace Operator

Let \(D\) be a domain in \(\mathbb{R}^n\), such that all excessive functions on \(D\) are purely excessive. Recall that the Laplace operator \(\Delta\) on \(D\) can be expressed in
terms of the infinitesimal generator of the Markov semigroup of \((X_t)\), more precisely, the infinitesimal generator of \((X_t)\) is \(\frac{\Delta}{2}\). For \(0 < \alpha < 1\) we can define, analytically, fractional power \((-\Delta)^\alpha\) on \(D\) (see, for example, section 2.6 in [2]). From the probabilistic point of view, we consider \(\alpha\)-subordinated Markov process, say \((X_t^\alpha)\), with respect to the Brownian motion killed upon exit from \(D\). The potential of \((X_t^\alpha)\) is exactly the \(V^\alpha\)-potential analyzed in the previous section. We claim that \((X_t^\alpha)\) and \(V^\alpha\) are natural choices to treat \((-\Delta)^\alpha\) on \(D\). The consequence is that the results of section 2 apply to the fractional power of \((-\Delta)\).

As usual, we use \(C_0^{\infty} = C_0^{\infty}(D)\) to denote the family of infinitely differentiable functions with compact support in \(D\).

**Lemma 2** If \(u\) is positive and \(V^{1-\alpha}u\) is not identically infinite, then for every \(\phi \in C_0^{\infty}\),

\[
\int_D u(x)[(-\Delta)^\alpha \phi](x)\,dx < \infty. \tag{26}
\]

**Proof.** Using formulae (6.9) and (6.15) in [2], pages 70-72, we obtain

\[
\int_D u(-\Delta)^\alpha\phi\,dx = \int_D u(-\Delta)^\alpha V^\alpha V^{1-\alpha} \Delta \phi\,dx
\]

\[
= \int_D u|V^{1-\alpha} \Delta \phi|\,dx
\]

\[
\leq \int_D u V^{1-\alpha} |\Delta \phi|\,dx
\]

\[
= \int_D (V^{1-\alpha}u)|\Delta \phi|\,dx < \infty,
\]

since \(|\Delta \phi| > 0\) only on a compact set and \(V^{1-\alpha}u\) is locally integrable, by Proposition 1.

**Q.E.D.**

The immediate consequence of this lemma is that, whenever the conditions of Lemma 2 are satisfied, we can perform the same computation as in the proof of Lemma 2 but without absolute values, to obtain the formula

\[
\int_D u(x)[(-\Delta)^\alpha \phi](x)\,dx = \int_D (V^{1-\alpha}u)(x)(\Delta \phi)(x)\,dx. \tag{27}
\]

The following theorem describes the relations between \(V^\alpha\)-harmonic functions and the solutions (in the sense of distributions) of the equation

\((-\Delta)^\alpha u = 0.\)
Theorem 4 Let $g$ be a positive $V^{\alpha}$-harmonic function such that $V^{1-\alpha}g$ is not identically infinite. Then, for every $\phi \in C_0^\infty$,

$$\int_D g \cdot [(-\Delta)^\alpha \phi] dx = 0. \tag{28}$$

Conversely, suppose that $g$ is a positive function such that $V^{1-\alpha}g$ is not identically infinite and such that (28) is satisfied for all $\phi \in C_0^\infty$. Then $g$ is $V^{\alpha}$-harmonic.

Proof. Suppose that $g$ is a positive $V^{\alpha}$-harmonic function and that $h = V^{1-\alpha}g$ is not identically infinite. It follows, by Theorem 2, that $h$ is a positive, locally integrable $U$-harmonic function. Thus

$$\int_D h(x)(\Delta \phi)(x) dx = 0, \tag{29}$$

for every $\phi \in C_0^\infty$ (see, for example, [4]). Since $g$ satisfies the conditions of Lemma 2, (28) follows immediately, by (27).

Suppose now that $g$ is a positive function such that (28) is true and such that $V^{1-\alpha}g$ is not identically infinite. Then $g$ satisfies the conditions of Lemma 2 and (28). Applying (27) we obtain that $h$ satisfies (29), i.e., $h$ is a positive $U$-harmonic function. By Theorem 2, $g$ is a $V^{\alpha}$-harmonic function.

Q.E.D.

We will finish this paper with the following straightforward consequence of Theorem 3 and Theorem 4.

Corollary 1 Suppose that $u$ is positive, $V^{1-\alpha}u$ is not identically infinite, and that, for every $\phi \in C_0^\infty$,

$$\int_D u[(-\Delta)^\alpha \phi] dx = 0. \tag{30}$$

Then $u$ is continuous on $D$.

References


