Quadratic Forms Corresponding to the Generalized Schrödinger Semigroups

J. Glover,* M. Rao, H. Šikić,† and R. Song‡

Department of Mathematics, 201 Walker Hall, University of Florida, Gainesville, Florida 32611

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Suppose that \(X_t\) is the standard Brownian motion in \(\mathbb{R}^d, d \geq 3\), that \(\rho \in H^1(\mathbb{R}^d)\) is a bounded continuous function such that \(|\nabla \rho|^2\) belongs to the Kato class and \(\mu\) is a measure belonging to the Kato class. Let \(A^p\) be defined as \(A^p = \rho(X_t) - \rho(X_0) - \int_0^t \nabla \rho(X_s) \cdot dX_s\), and let \(A^\mu\) be the continuous additive functional with \(\mu\) as its Revuz measure. Define \(A\) as the sum of the two additive functionals above. Then the semigroup defined as

\[
T_t f(x) = E^x \{ e^{\delta f(X_t)} \}
\]

is called a generalized Schrödinger semigroup. In this paper we identify the quadratic form corresponding to \((T_t, H^1(\mathbb{R}^d))\) with

\[
\mathcal{G}(u, v) = \frac{1}{2} \int \nabla u(x) \cdot \nabla v(x) \, dx + \int \nabla \rho(x) \cdot \nabla \rho(x) \, dx - \int \rho(x) \mu(dx).
\]


INTRODUCTION

Perturbation of Brownian motion created by multiplicative functionals is a classical topic dating back to the early researches of Mark Kac. This subject has been revived in the past 10 years by both probabilists and mathematical physicists interested in perturbations \(\mathcal{A} + T\) of the Laplace operator \(\mathcal{A}\). Initially, the case when \(T\) is multiplication by a function \(q\) was the focus of attention, resulting in the gauge and supergauge theorems (see, for instance, [3, 7, 8, 18]). These results were subsequently generalized to the case when \(T\) is multiplication by a measure (see, for instance, [2, 4, 5]).

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† Department of Mathematics, University of Zagreb, 41000, Zagreb, Croatia.

‡ Current address: Department of Mathematics, Northwestern University, Evanston, IL 60208.

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The case when $T$ is a distribution has also been examined. See, for instance, [1] and the references therein; for more recent development, one can see [12, 13].

Let $X_t$ be the Brownian motion in $\mathbb{R}^d$, $d \geq 3$. Let $\mu$ be a measure in the Kato class with associated additive functional $\overline{A}_t$ (i.e., $\mu$ is the Revuz measure of $\overline{A}_t$). Let $\rho \in H^1(\mathbb{R}^d)$ be bounded and continuous with $|\nabla \rho|^2$ in the Kato class, and define

$$A_{t}^{[\rho]} = \rho(X_t) - \rho(X_0) - \int_0^t \nabla \rho(X_s) \cdot dX_s.$$ 

Put

$$A_t = \overline{A}_t + A_{t}^{[\rho]}.$$ 

In [13], the properties of the Schrödinger semigroup

$$T_t f(x) = E^x \{ e^{A_t} f(X_t) \}$$

were studied in detail. It was proven, among other things, that $(T_t)_{t \geq 0}$ is a strongly continuous semigroup on $L^2(\mathbb{R}^d)$. When the additive functional $A_t$ is of bounded variation with its Revuz measure belonging to the Kato class, the infinitesimal generator of $(T_t)_{t \geq 0}$, or equivalently, the quadratic form associated with $(T_t)_{t \geq 0}$, is well known. In our case, however, we are perturbing Brownian motion by an additive functional which is generally not of bounded variation, not even a semimartingale. To the best of our knowledge, no one has written down explicitly the infinitesimal generator of $(T_t)_{t \geq 0}$, or equivalently, the quadratic form associated with $(T_t)_{t \geq 0}$ yet.

The main task of this article is to identify the quadratic form associated with $T_t$. For $u$ and $v$ in $H^1(\mathbb{R}^d)$, define

$$\mathcal{F}(u, v) = \frac{1}{2} \int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla v(x) \, d(x)$$

$$+ \int_{\mathbb{R}^d} \nabla (uv)(x) \cdot \nabla \rho(x) \, dx - \int_{\mathbb{R}^d} u(x) \, v(x) \, \mu(dx).$$

Then, $(\mathcal{F}, H^1(\mathbb{R}^d))$ is the closed quadratic form corresponding to $T$, (see Theorem (3.3)). On the way to proving this, we improve a result of Albeverio and Ma [2] (see Theorem (2.6)). They examined the case when $\rho = 0$ and showed that $\mathcal{F}$ is the quadratic form associated with $T_t$, but they identified the domain of $\mathcal{F}$ as $H^1(\mathbb{R}^d) \cap L^2(\mu^*)$ with $\mu^*$ being the total variation of $\mu$. We show that $H^1(\mathbb{R}^d) \subset L^2(\mu^*)$, simplifying their result. Much of our analysis relies on a remarkable lemma (see Lemma (2.1)) which we have never seen in the literature, but which deserves to be widely
known: if $\lambda$ is a nonnegative Radon measure, then there exists an $M_d > 0$ depending only on the dimension $d$ such that for any quasicontinuous $u \in H^1(R^d)$,
\[
\int_{R^d} u^2(x) \lambda(dx) \leq M_d \left( \sup_{x \in R^d} \int_{R^d} \frac{\lambda(dy)}{|x-y|^{d-2}} \right) \times \left( \int_{R^d} |\nabla u|^2(x) \, dx + \int_{R^d} u^2(x) \, dx \right).
\]

Although the results of this article are stated for the case of $R^d$, the proofs can be easily modified to treat the perturbation of Brownian motion killed upon leaving any open subset of $R^d$.

1. Preliminaries

In this paper, we shall always assume that $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, P^x)$ is a standard Brownian motion in $R^d$, $d \geq 3$, and that $(\mathcal{E}, H^1(R^d))$ is the closed Dirichlet form associated with $X$. The Sobolev space $H^1(R^d)$ is defined by
\[
H^1(R^d) = \left\{ u \in L^2(R^d) : \frac{\partial u}{\partial x_i} \in L^2(R^d), i = 1, 2, ..., d \right\}
\]
and $\mathcal{E}$ is defined by
\[
\mathcal{E}(u, v) = \frac{1}{2} \int_{R^d} \nabla u \cdot \nabla v \, dx, \quad \forall u, v \in H^1(R^d).
\]

For a positive number $\alpha$, we shall write
\[
\mathcal{E}_\alpha(u, v) = \mathcal{E}(u, v) + \alpha(u, v),
\]
where $(u, v)$ denotes the inner product in $L^2(R^d)$.

In this paper, we make the convention that whenever we talk about an element in $H^1(R^d)$, we mean the quasicontinuous version of that element.

As usual we use $p(t, x, y)$ to denote the transition density of $X_t$. We shall sometimes write
\[
P_t \mu(x) = \int_{R^d} p(t, x, y) \mu(dy)
\]
and
\[
G_\alpha \mu(x) = \int_0^\infty e^{-\alpha t} P_t \mu(x) \, dt
\]
provided the right-hand sides make sense.
If $\rho \in H^1(R^d)$ is bounded and continuous, then

$$A_t^{[\rho]} = \rho(X_t) - \rho(X_0) - \int_0^t \nabla \rho(X_s) \cdot dX_s$$

is a continuous additive functional of zero energy with respect to $X$. In general, $A_t^{[\rho]}$ is not of bounded variation. We shall call $(A_t^{[\rho]})$ the continuous additive functional of zero energy generated by $\rho$.

(1.1) Definition. A signed Radon measure $\mu$ on $R^d$ is said to be in the Kato class if

$$\lim \sup_{r \downarrow 0} \int_{|x - y| < r} \frac{\mu^*(dy)}{|x - y|^{d-2}} = 0,$$

where $\mu^* = \mu^+ + \mu^-$ with $\mu^+$ and $\mu^-$ being the positive part and the negative part of $\mu$, respectively. In the sequel we shall use $K_d$ to denote the Kato class.

It is well known (see, for instance, [4] or [12]) that for any $\mu \in K_d$ there exists a unique continuous additive functional $A_t$ of $X$ with $\mu$ as its Revuz measure.

(1.2) Definition. A bounded continuous function $\rho \in H^1(R^d)$ is said to be admissible if $|\nabla \rho|^2 (x) \, dx$ is in the Kato class $K_d$.

The connection between measures in the Kato class and admissible functions was given in [12]. It is proven there that if $\mu \in K_d$ is of compact support and that if $(A_t)$ is the continuous additive functional associated with $\mu$, then there is an admissible $\rho$ such that $A_t = A_t^{[\rho]}$.

From now on, we are going to fix an admissible function $\rho$ and a measure $\mu \in K_d$. We will use $A_t$ to denote the continuous additive functional whose Revuz measure is $\mu$, $A_t$ to denote the sum $A_t^{[\rho]} + A_t$, and $(T_t)_{t > 0}$ to denote the following generalized Schrödinger semigroup:

$$T_t f(x) = E^x \{ e^{A_t f(X_t)} \}.$$

Put

$$J(x) = \begin{cases} k \exp \left( \frac{-1}{1 - |x|^2} \right), & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where $k$ is a positive constant such that

$$\int_{R^d} J(x) \, dx = 1.$$
For any positive integer \( n \), we define

\[
J_n(x) = n^d J(nx),
\]
\[
\tilde{\rho}_n(x) = I_n(x) \cdot \rho(x),
\]
\[
\rho_n(x) = J_n * \tilde{\rho}_n(x),
\]

where

\[
I_n(x) = \begin{cases} 
1, & |x| < n, \\
\exp \left( \frac{-1}{1 - (|x| - n)^2} \right), & n \leq |x| < n + 1, \\
0, & |x| \geq n + 1.
\end{cases}
\]

Then the following result is proved in [13].

(1.3) **Proposition.**  (1) For any \( n \geq 1 \),

\[
\|\rho_n\|_{\infty} \leq \|\rho\|_{\infty};
\]

(2) For any \( n \geq 1 \), \( \rho_n \in C^\infty_\circ (\mathbb{R}^d) \), i.e., \( \rho_n \) is an infinitely differentiable function with compact support;

(3) \( \rho_n \) converges to \( \rho \) uniformly on any compact subset of \( \mathbb{R}^d \);

(4) \( \rho_n \) converges to \( \rho \) in \( H^1(\mathbb{R}^d) \);

(5)

\[
\limsup_{\delta \downarrow 0} \sup_{n, \lambda} \left[ \frac{\int_{|x - y| \leq \delta} |\nabla \rho_n|^2(y)}{|x - y|^{d-2}} \, dy \right] = 0.
\]

(6)

\[
\limsup_{n \to \infty} \sup_{x} \int_{\mathbb{R}^d} \frac{|\nabla (\rho_n - \rho)|^2(y)}{|x - y|^{d-2}} \, dy = 0.
\]

If for any \( n \geq 1 \) we define

\[
A^{(n)}_t = A^\rho_{\pm} + \Delta_t,
\]
\[
T^{(n)}_t f(x) = E^x \left\{ e^{A^{(n)}_t} f(X_t) \right\},
\]

then we have the following result.

(1.4) **Proposition.**  (1) For any \( n \geq 1 \), \( T^{(n)}_t \) has a symmetric continuous integral kernel \( q^{(n)}(t, x, y) \). Furthermore, there exist two positive constants \( C \) and \( \beta' \) independent of \( n \) such that

\[
q^{(n)}(t, x, y) \leq Ce^{\beta' t} t^{-d/2}.
\]
(2) $T_t$ also has a symmetric continuous integral kernel $q(t, x, y)$ such that

$$q(t, x, y) \leq Ce^{\beta' t - d/2},$$

where $C$ and $\beta'$ are the constants specified in (1).

(3) $(T_t)_{t > 0}$ is a strongly continuous semigroup on $L^2(R^d)$.

(4) For any $f \in L^2(R^d)$, any $t > 0$ and any $x \in R^d$, $T_t^{(n)} f(x) \rightarrow T_t f(x)$.

(5) For any $f \in L^2(R^d)$ and any $t > 0$, $T_t^{(n)} f \rightarrow T_t f$ strongly in $L^2(R^d)$.

**Proof.** (1), (2), (3), and (4) are proved in [13], so we need only to prove (5).

From Lemma (2.4) of [13] we know that there exist constants $C_1$ and $\beta_1$ such that

$$\sup_{n, x} E^x \{ e^{2A_t^{(n)}} \} \leq C_1 e^{2\beta_1 t},$$

therefore for any $t > 0$ and any $x \in R^d$,

$$|T_t^{(n)} f(x)| \leq \sup_{n, x} (E^x \{ e^{2A_t^{(n)}} \})^{1/2} (E^x f^2(X_t))^{1/2} \leq C_1 e^{\beta_1 t} (E^x f^2(X_t))^{1/2}.$$

Thus (5) follows easily from (4) by the dominated convergence theorem.

**Remark.** The above estimates for the densities of Schrödinger semigroups are perhaps derivable from the results of Chap. 2 of [10]. However, such a derivation is not all that clear so a direct proof is provided here.

From the above result we can see that for any $\alpha > \beta$ and any $f \in L^2(R^d)$,

$$R_\alpha^{(n)} f(x) = \int_0^\infty e^{-\alpha t} T_t^{(n)} f(x) \, dt$$

$$R_\alpha f(x) = \int_0^\infty e^{-\alpha t} T_t f(x) \, dt$$

are square integrable functions on $R^d$. Furthermore, from the above result we know that for any $f \in L^2(R^d)$, $R_\alpha^{(n)} f$ converges to $R_\alpha f$ strongly in $L^2(R^d)$.

2. **The Quadratic Form**

In this paper, we are going to deal frequently with integrals of the type

$$\int_{R^d} u^2(x) \lambda(dx),$$

where $u$ is in $H^1(R^d)$ and $\lambda$ is a positive measure. The following lemma provides an estimate on the above integral.
(2.1) Lemma. There exists an \( M_d > 0 \) depending only on the dimension \( d \) such that for any \( u \in H^1(\mathbb{R}^d) \), \( u \neq 0 \), and any nonnegative Radon measure \( \lambda \),

\[
\int_{\mathbb{R}^d} u^2(x) \lambda(dx) \leq M_d \left( \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\lambda(dy)}{|x-y|^{d-2}} \right) \left( \int_{\mathbb{R}^d} |\nabla u|^2(x) \, dx + \int_{\mathbb{R}^d} u^2(x) \, dx \right).
\]

Proof. Put

\[
k_1(x, y) = \int_0^{\infty} e^{-t} p(t, x, y) \, dt
\]

and let \( K_1 \) denote the following operator:

\[
K_1 f(x) = \int_{\mathbb{R}^d} k_1(x, y) f(y) \, dy.
\]

Without loss of generality, we can assume that \( \lambda \) does not charge polar sets, since otherwise the right-hand side of the inequality is infinite. (See, for instance, p. 224 of [6].) Since \( \lambda \) does not charge polar sets, we know that for every Borel set \( A \),

\[
1_A(x) \leq P^x(T_A = 0), \quad \lambda\text{-a.e.},
\]

where \( T_A \) is the first hitting time of \( A \) by the Brownian motion killed with exponential rate 1.

For any set \( A \), let \( v_A \) be the equilibrium measure of \( A \). Using properties of the capacity, we can get

\[
\int_{\mathbb{R}^d} u^2(x) \lambda(dx) = 2 \int_0^{\infty} t \lambda(|u| \geq t) \, dt
\]

\[
= 2 \int_0^{\infty} t \int_{\mathbb{R}^d} 1_{(|u| \geq t)}(x) \lambda(dx) \, dt
\]

\[
\leq 2 \int_0^{\infty} t \int_{\mathbb{R}^d} P^x(T_{|u| \geq t} = 0) \lambda(dx) \, dt
\]

\[
\leq 2 \int_0^{\infty} t \int_{\mathbb{R}^d} P^x(T_{|u| < \zeta} = 0) \lambda(dx) \, dt
\]

\[
= 2 \int_0^{\infty} t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k_1(x, y) v_{|u| \geq t}(dy) \lambda(dx) \, dt
\]

\[
\leq 2 \left( \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} k_1(x, y) \lambda(dy) \right) \int_0^{\infty} t C_1(|u| \geq t) \, dt
\]

\[
\leq 2 M'_d \left( \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\lambda(dy)}{|x-y|^{d-2}} \right) \int_0^{\infty} t C_1(|u| \geq t) \, dt,
\]
where $\zeta$ is the life time of the Brownian motion killed with exponential rate 1, $C_1$ is the capacity of this killed Brownian motion, and $M'_d$ is a constant depending only on $d$.

From Thm. 1.6 of [14], we get that

$$\int_0^\infty tC_1(|u| \geq t)\, dt \leq M'_d \cdot \inf\{(K_1v, v)\},$$

where $M'_d$ depends only on $d$ and the infimum is taken over all nonnegative measures $v$ such that $K_1v \geq |u|$ up to a set of zero $C_1$ capacity. The infimum above can be replaced by infimum over all measures $v$ such that $K_1v \geq |u|$ a.e., since, by our convention, $u$ is quasicontinuous. Therefore, to finish the proof it suffices to show that there exists a $p \in H^1(R^d)$ of the form $K_1v$ for some positive measure $v$ such that

$$p \geq |u| \text{ a.e. and } \|p\|_{H^1} \leq \|u\|_{H^1}.$$

Consider now the set $\mathcal{P}$ of potentials of finite energy:

$$\mathcal{P} = \{K_1v: v \text{ is a positive measure and } (K_1v, v) < \infty\}.$$

Following the proof on pp. 97–99 of [16] for Newtonian potentials, we can prove the analogous result for $K_1$ to get that every $K_1v \in \mathcal{P}$ is in $H^1(R^d)$ and satisfies

$$\|K_1v\|_{H^1} = (K_1v, v)^{1/2}.$$

Hence $\mathcal{P}$ is a closed, convex subset of $H^1(R^d)$, since the set of positive measures of finite energy is complete in the energy norm (see, for instance, [17]). It follows that there exists a projection $p$ of $|u|$ on the set $\mathcal{P}$. Using the standard manipulation by quadratic equations (see, for example, Chap. 7 of [17]), we can prove that

$$(p - |u|, q)_{H^1} \geq 0, \quad \forall q \in \mathcal{P},$$

and

$$(p - |u|, p)_{H^1} \leq 0.$$  

Because $(\phi, K_1v)_{H^1} = (\phi, v)$, it follows that $p \geq |u|$ a.e. and that $(|u| - p, p)_{H^1} = 0$. Hence

$$(p, p)_{H^1} = (p, |u|)_{H^1} \leq \|p\|_{H^1} \|u\|_{H^1},$$

which finishes the proof.

Remark. The assertion of the above lemma is trivially fulfilled when $u$ is the zero element of $H^1(R^d)$ if we take the version of $u$ which is identically zero on $R^d$. 
The following result is elementary. Since we cannot find a reference for it, we provide a proof here for the reader's convenience.

(2.2) Lemma (Covering Lemma). For every $r > 0$, there exists a sequence $\{c_i : i \geq 1\}$ of points in $R^d$ such that

1. $\{B(c_i, 3r) : i \geq 1\}$ covers $R^d$,

2. for every $x \in R^d$, $B(x, r)$ intersects at most $5^d$ balls from $\{B(c_i, 3r) : i \geq 1\}$ and $B(x, r)$ intersects at most $8^d$ balls from $\{B(c_i, 6r) : i \geq 1\}$, where for any $c \in R^d$ and any $\delta > 0$, $B(c, \delta)$ stands for the open ball of radius $\delta$ around $c$.

Proof. Let us start with a countable cover of $R^d$ by balls of radius $r$ and let us denote this cover by $\{B_i = B(a_i, r) : i \geq 1\}$. We can take a maximal subset of $\{B_i\}$ such that any two balls in the subset are disjoint. First, we take $i_1 = 1$. Then we take $i_2 = \min\{i > i_1 : B_i \cap B_{i_1} = \emptyset\}$ and we continue inductively in the same way. Therefore, we get a sequence $\{B_k : k \in N\}$ with the following two properties:

1. $k \neq l \Rightarrow B_k \cap B_l = \emptyset$;

2. For any $i \in N$ there exists an $i_k \geq 1$ such that $B_i$ and $B_{i_k}$ have a nonempty intersection.

Obviously $\{i_k : k \geq 1\}$ must be infinite. We claim that $c_k = a_{i_k}$ satisfies all the requirements of this lemma.

Let $x \in R^d$. Since $\{B_i\}$ is a cover, there exists $B_i$ such that $x \in B_i$. By (2) above, there exists $B_{i_k}$ such that $B_i \cap B_{i_k} \neq \emptyset$. Hence,

\[
|x - c_k| = |x - a_{i_k}|
\leq |x - a_i| + |a_i - a_{i_k}|
< r + 2r = 3r.
\]

It follows that $\{B(c_k, 3r) : k \geq 1\}$ is a cover of $R^d$.

Let $x \in R^d$ and let $B(x, r)$ intersect $B(c_k, 3r)$. Then $|x - c_k| < 4r$; i.e., $B(c_k, r)$ is contained in $B(x, 5r)$. Since the balls in $\{B(c_i, r) : i \geq 1\}$ are pairwise disjoint, there can be no more than

\[
\frac{m(B(x, 5r))}{m(B(c_i, r))}
\]

balls of the form $B(c_i, r)$ inside $B(x, 5r)$, where $m$ denote the Lebesgue measure in $R^d$. Since

\[
m(B(x, 5r)) = \frac{(5r)^d \pi^{d/2}}{d(d/2 + 1)}
\]
and
\[ m(B(c_i, r)) = \frac{r^d \pi^{d/2}}{\Gamma(d/2 + 1)}, \]
there are no more than \( 5^d \) balls of the form \( B(c_i, 3r) \) intersecting \( B(x, r) \).

The second conclusion of the lemma can be proved by starting with the radius \( 6r \) in the paragraph above and going over the same argument.

Using the two lemmas above, we can get the following theorems.

**Theorem (2.3)** If \( \lambda \in K_d \) is nonnegative, then for any \( u \in H^1(R^d) \),
\[ \int_{R^d} u^2(x) \lambda(dx) < 0. \]

Furthermore, for every \( \varepsilon > 0 \), there exists a \( C(\varepsilon) > 0 \) such that for every \( u \in H^1(R^d) \),
\[ \int_{R^d} u^2(x) \lambda(dx) \leq \varepsilon \int_{R^d} |\nabla u|^2 \lambda(x) dx + C(\varepsilon) \int_{R^d} u^2(x) dx. \]

**Proof.** Fix an arbitrary \( \varepsilon > 0 \). Since \( \lambda \in K_d \), there exists a \( \delta > 0 \) such that
\[ \sup_{x \in R^d} \int_{|x-y| < \delta} \frac{\lambda(dy)}{|x-y|^{d-2}} < \varepsilon. \]

Let \( r = \delta/6 \) and let \( \{c_i : i \geq 1\} \) be the corresponding sequence obtained in the covering lemma. Let \( \varphi : R^d \rightarrow [0, 1] \) be an infinitely differentiable function such that \( \varphi(x) \equiv 1 \) on \( B(0, 3r) \) and \( \varphi(x) \equiv 0 \) outside of \( B(0, 6r) \). For every \( i \geq 1 \), we denote by \( \varphi_i \) the following translate of \( \varphi \): \( \varphi_i(x) = \varphi(x - c_i) \).

Note that there exists a constant \( M = M(\varepsilon) > 0 \) such that \( |\nabla \varphi_i|^2 \leq M \) for every \( i \). For the sake of simplicity, we denote \( B(c_i, 3r) \) by \( B \) and \( B(c_i, 6r) \) by \( B' \).

Since \( \{B_i : i \geq 1\} \) is a cover of \( R^d \), and \( 1_{B_i} = 1_{B'} \varphi_i^2(x) \), we know that
\[ \int_{R^d} u^2(x) \lambda(dx) \leq \sum_{i=1}^{\infty} \int_{B} 1_{B_i} (x) u^2(x) \lambda(dx) = \sum_{i=1}^{\infty} \int_{B} u^2(x) \varphi_i^2(x) \lambda(dx). \]

Note that \( u \varphi_i \in H^1(R^d) \), so we can apply Lemma (2.1) to every integral under the above sum:
\[ \int_{B_i} u^2(x) \varphi_i^2(x) \lambda(dx) \leq M_d \left( \sup_{x \in R^d} \int_{B_i} \frac{\lambda(dy)}{|x-y|^{d-2}} \right) \left( \int_{R^d} (u \varphi_i)^2 (x) dx + \int_{R^d} |\nabla (u \varphi_i)|^2 (x) dx \right). \]
Recall that the Newtonian potential satisfies the maximum principle; therefore,
\[
\sup_{x \in \mathbb{R}^d} \int_{B_r(x)} \frac{\lambda(dy)}{|x-y|^{d-2}} = \sup_{x \in \mathbb{B}_r} \int_{B_r(x)} \frac{\lambda(dy)}{|x-y|^{d-2}} \leq \sup_{x \in \mathbb{B}_r} \int_{|x-y| < \delta} \frac{\lambda(dy)}{|x-y|^{d-2}} < \varepsilon.
\]

It follows that
\[
\int_{B_r} u^2(x) \varphi_i^2(x) \lambda(dx) \leq M_d \varepsilon \left( \int_{\mathbb{R}^d} u^2(x)(\varphi_i^2 + 2|\nabla \varphi_i|^2)(x) \, dx + \int_{\mathbb{R}^d} |\nabla u|^2(x)(2\varphi_i^2(x)) \, dx \right),
\]
which implies
\[
\int_{\mathbb{R}^d} u^2(x) \lambda(dx) \leq M_d \varepsilon \left( \int_{\mathbb{R}^d} u^2(x) \left( \sum_{i=1}^{\infty} \varphi_i^2 + 2 \sum_{i=1}^{\infty} |\nabla \varphi_i|^2 \right)(x) \, dx 
+ \int_{\mathbb{R}^d} |\nabla u|^2(x) \left( 2 \sum_{i=1}^{\infty} \varphi_i^2 \right)(x) \, dx \right).
\]

For every $x \in \mathbb{R}^d$, $B(x, r)$ intersects at most $5^d$ balls from $\{B_i\}$ and at most $8^d$ balls from $\{B'_i\}$. Thus, for any $x \in \mathbb{R}^d$,
\[
\sum_{i=1}^{\infty} \varphi_i^2(x) \leq 8^d,
\]
\[
\sum_{i=1}^{\infty} |\nabla \varphi_i|^2(x) \leq M \cdot 8^d.
\]

Hence, we obtain that
\[
\int_{\mathbb{R}^d} u^2(x) \lambda(dx) \leq 2M_d 8^d \varepsilon \int_{\mathbb{R}^d} |\nabla u|^2(x) \, dx + M_d 8^d \varepsilon (1 + 2M(\varepsilon)) \int_{\mathbb{R}^d} u^2(x) \, dx.
\]

The proof is now complete.

(2.4) Theorem. For any $\varepsilon > 0$, there exists a $C(\varepsilon) > 0$ such that for any $u \in H^1(\mathbb{R}^d)$,
\[
\left| \int_{\mathbb{R}^d} u(x) \nabla u(x) \cdot \nabla \varphi(x) \, dx \right| < \varepsilon \int_{\mathbb{R}^d} |\nabla u|^2(x) \, dx + C(\varepsilon) \int_{\mathbb{R}^d} u^2(x) \, dx.
\]
Proof. Using the Schwarz inequality, we can get

\[ \left| \int_{\mathbb{R}^d} u(x) \nabla u(x) \cdot \nabla \rho(x) \, dx \right| \leq \left( \int_{\mathbb{R}^d} u^2(x) |\nabla \rho|^2(x) \, dx \right)^{1/2} \left( \int_{\mathbb{R}^d} |\nabla u|^2(x) \, dx \right)^{1/2}. \]

However, by Theorem (2.4) we know that for any \( \varepsilon > 0 \), there exists \( C(\varepsilon) > 0 \) such that for any \( u \in H^1(\mathbb{R}^d) \),

\[ \int_{\mathbb{R}^d} u^2(x) |\nabla \rho|^2(x) \, dx < \varepsilon \int_{\mathbb{R}^d} |\nabla u|^2(x) \, dx + C(\varepsilon) \int_{\mathbb{R}^d} u^2(x) \, dx. \]

Thus,

\[ \left| \int_{\mathbb{R}^d} u(x) \nabla u(x) \cdot \nabla \rho(x) \, dx \right| \leq \left( \varepsilon \int_{\mathbb{R}^d} |\nabla u|^2(x) \, dx + C(\varepsilon) \int_{\mathbb{R}^d} u^2(x) \, dx \right)^{1/2} \left( \int_{\mathbb{R}^d} u^2(x) \, dx \right)^{1/2}. \]

Since for any \( a > 0 \) and any nonnegative numbers \( A, b, B \) we have

\[
A^{1/2} (aA + bB)^{1/2} = a^{1/2} A^{1/2} \left( A + \frac{b}{a} B \right)^{1/2} \leq a^{1/2} \left( A + \frac{b}{a} B \right)^{1/2} \left( A + \frac{b}{a} B \right)^{1/2} = a^{1/2} \left( A + \frac{b}{a} B \right),
\]

it follows that

\[ \left| \int_{\mathbb{R}^d} u(x) \nabla u(x) \cdot \nabla \rho(x) \, dx \right| \leq \varepsilon^{1/2} \int_{\mathbb{R}^d} |\nabla u|^2(x) \, dx + \frac{C(\varepsilon)}{\varepsilon^{1/2}} \int_{\mathbb{R}^d} u^2(x). \]

The proof is now complete.

From Theorems (2.3) and (2.4), we know that \( \mathcal{E} \) given by

\[ \mathcal{E}(u, v) = \mathcal{S}(u, v) + \int_{\mathbb{R}^d} \nabla(uv)(x) \cdot \nabla \rho(x) \, dx - \int_{\mathbb{R}^d} u(x) v(x) \mu(dx) \]
is finite for all $u, v \in H^1(R^d)$ and that $(\tilde{\mathcal{E}}, H^1(R^d))$ is a quadratic form. Furthermore, we know that for any $\varepsilon > 0$, there exists a $C(\varepsilon) > 0$ such that for any $u \in H^1(R^d)$,

$$|((\tilde{\mathcal{E}} - \mathcal{E})(u, u)| < \varepsilon \int_{R^d} |\nabla u|^2 (x) \, dx + C(\varepsilon) \int_{R^d} u^2(x);$$

i.e., the relative form bound of $\tilde{\mathcal{E}} - \mathcal{E}$ with respect to $\mathcal{E}$ is zero. Therefore, by Thm. 4.23 of [9], we get the following main result of this section.

(2.5) Theorem. $(\tilde{\mathcal{E}}, H^1(R^d))$ is a lower semibound, closed quadratic form.

From this result we know that there is a unique lower semibounded selfadjoint operator $H$ corresponding to the form $(\tilde{\mathcal{E}}, H^1(R^d))$. In the next section, we will prove that $(\tilde{\mathcal{E}}, H^1(R^d))$ is the quadratic form corresponding to the generalized Schrödinger semigroup $(T_r)_{t \geq 0}$; i.e., $H$ is the generator of the generalized Schrödinger semigroup $(T_r)_{t \geq 0}$.

In [2], Albeverio and Ma proved that if we define

$$\mathcal{E}_\mu(u, v) = \mathcal{E}(u, v) - \int_{R^d} u(x) v(x) \mu(dx),$$

then $(\mathcal{E}_\mu, H^1(R^d) \cap L^2(\mu^*))$ is the quadratic form corresponding to the following Schrödinger semigroup:

$$P^\mu_t f(x) = E^\mu \{ e^{\lambda f(X_t)} \}.$$

From Theorem (2.3), we know that

$$H^1(R^d) \cap L^2(\mu^*) = H^1(R^d),$$

thus we have the following result which is an improvement of the corresponding result in [2].

(2.6) Theorem. $(\mathcal{E}_\mu, H^1(R^d))$ is the quadratic form corresponding to the Schrödinger semigroup $(P^\mu_t)$.

3. The Connection Between $\tilde{\mathcal{E}}$ and $(T_t)$

Let $-\beta''$ be a lower bound of $\tilde{\mathcal{E}}$; i.e., $\beta''$ is a number such that for every $u \in H^1(R^d)$,

$$\tilde{\mathcal{E}}(u, u) \geq -\beta''(u, u).$$
In the remainder of this section, $\beta$ will always stand for the number $\beta' \lor \beta''$, where $\beta'$ is specified by (1.4).

(3.1) Lemma. For any $\alpha > \beta$ and any $f \in L^2(R^d)$, $\{R^{(n)}_x f : n \geq 1\}$ is bounded in $H^1(R^d)$.

Proof. For any $n \in N$, put $v_n = R^{(n)}_x f$. From Proposition (1.4), we know that $v_n$ converge strongly in $L^2(R^d)$, thus $\|v_n\| \leq C_1$ for some constant $C_1 > 0$. Since for any $n \in N$, $v_n$ is a weak solution to the equation

$$\frac{1}{2} \Delta u + (\beta \rho_n)u + u\mu - \alpha u + f = 0,$$

we know that for any $n \geq 1$,

$$\frac{1}{2} \int_{R^d} |\nabla v_n|^2(x) \, dx$$

$$= -\int_{R^d} \nabla \rho_n(x) \cdot \nabla v_n^2(x) \, dx + \int_{R^d} v_n^2(x) \mu(dx)$$

$$- \alpha \int_{R^d} v_n^2(x) \, dx + \int f(x) v_n(x) \, dx;$$

consequently,

$$\int_{R^d} |\nabla v_n|^2(x) \, dx \leq 2 \left[ \int_{R^d} \nabla \rho_n(x) \cdot \nabla v_n^2(x) \, dx \right] + 2 \int_{R^d} v_n^2(x) \mu^*(dx)$$

$$2 \alpha \int_{R^d} v_n^2(x) \, dx + 2 \|f\|_2 \|v_n\|_2.$$

From Theorems (2.3) and (2.4), we know that there exists a constant $C_2 > 0$ such that

$$2 \left[ \int_{R^d} \nabla \rho_n(x) \cdot \nabla v_n^2(x) \, dx \right] + 2 \int_{R^d} v_n^2(x) \mu^*(dx)$$

$$\leq \frac{1}{2} \int_{R^d} |\nabla v_n|^2(x) \, dx + C_2 \|v_n\|_2^2,$$

thus

$$\int_{R^d} |\nabla v_n|^2(x) \, dx \leq \frac{1}{2} \int_{R^d} |\nabla v_n|^2(x) \, dx + (C_2 + 2\alpha) \|v_n\|_2^2 + \|f\|_2 \|v_n\|_2$$

$$\leq 2(C_2 + 2\alpha) C_1^2 + C_1 \|f\|_2.$$

The lemma is proved.
The following lemma is easy but not well known. We give the proof here for the readers’ convenience. In this lemma, $D_i$ stands for the generalized partial derivative $\partial / \partial x_i$.

(3.2) **Lemma.** Suppose that $\{u_n\} \subset H^1(R^d)$ and that $u_n \to u$ in $L^2(R^d)$. If for some $1 \leq i \leq d$, $\{D_i u_n\}$ is bounded in $L^2(R^d)$, then $D_i u \in L^2(R^d)$ and $D_i u_n$ converge weakly to $D_i u$ in $L^2(R^d)$.

**Proof.** For any $\psi \in C_0^\infty (R^d)$,

$$\int_{R^d} (D_i u_n)(x) \psi(x) \, dx = - \int_{R^d} u_n(x)(D_i \psi)(x) \, dx,$$

thus

$$\lim_{n \to \infty} \int_{R^d} (D_i u_n)(x) \psi(x) \, dx = - \int_{R^d} u(x)(D_i \psi)(x) \, dx.$$

On the other hand, however, $\{D_i u_n\}$ is bounded in $L^2(R^d)$, therefore there is a subsequence $\{D_i u_{n_k}\}$ which converges weakly in $L^2(R^d)$ to some function $f_i \in L^2(R^d)$; in particular,

$$\lim_{k \to \infty} \int_{R^d} (D_i u_{n_k})(x) \psi(x) \, dx = \int_{R^d} f_i(x) \psi(x) \, dx.$$

Thus, for any $\psi \in C_0^\infty$,

$$\int_{R^d} f_i(x) \psi(x) \, dx = - \int_{R^d} u(x)(D_i \psi)(x) \, dx;$$

that is to say, $D_i u$ exists and $D_i u = f_i \in L^2(R^d)$.

To prove that $\{D_i u_n\}$ converges weakly to $D_i u$ in $L^2(R^d)$, we need only to show that for any $g \in L^2(R^d),$

$$\lim_{n \to \infty} \int_{R^d} (D_i u_n)(x) g(x) \, dx = \int_{R^d} (D_i u)(x) g(x) \, dx.$$

For any $\varepsilon > 0$, take a $\psi \in C_0^\infty (R^d)$ such that $\|\psi - g\|_2 < \varepsilon$. Then

$$\left| \int_{R^d} (D_i u_n)(x) g(x) \, dx - \int_{R^d} (D_i u)(x) g(x) \, dx \right| \leq \int_{R^d} (D_i u_n)(x)(g - \psi)(x) \, dx + \int_{R^d} (D_i u_n)(x) \psi(x)$$

$$\quad - \int_{R^d} (D_i u)(x) \psi(x) \, dx + \int_{R^d} (D_i u)(x)(\psi - g)(x) \, dx \leq C \|g - \psi\|_2 + \int_{R^d} (D_i u_n)(x) \psi(x) - \int_{R^d} (D_i u)(x) \psi(x) \, dx,$$
where
\[ C = \| D_i u \|_2 \vee \sup_n \{ \| D_i u_n \|_2 \}. \]

Therefore there exists an $N > 0$ such that
\[ \left| \int_{\mathbb{R}^d} (D_i u_n)(x) g(x) \, dx - \int_{\mathbb{R}^d} (D_i u)(x) g(x) \, dx \right| < (C + 1) \varepsilon, \]
whenever $n > N$. The proof is now complete.

Now here is the main result of this section.

\( (3.3) \) Theorem. \((\mathcal{G}, H^1(\mathbb{R}^d))\) is the closed quadratic form corresponding to the generalized Schrödinger semigroup \((T_t)\).

Proof. We need only to show that for any $\alpha > \beta$ and any $f \in L^2(\mathbb{R}^d)$, $R_\alpha f$ is in $H^1(\mathbb{R}^d)$ and that
\[ \mathcal{E}_\alpha(R_\alpha f, v) = (f, v) \quad \forall v \in H^1(\mathbb{R}^d), \quad (3.4) \]
where for any $u, v \in H^1(\mathbb{R}^d)$,
\[ \mathcal{E}_\alpha(u, v) = \mathcal{G}(u, v) + \alpha(u, v). \]

The fact that for any $\alpha > \beta$ and any $f \in L^2(\mathbb{R}^d)$, $R_\alpha f$ is in $H^1(\mathbb{R}^d)$ follows directly from Lemmas (3.1) and (3.2). From Theorem (2.6), we know that for any $n \geq 1$, the quadratic form \((\mathcal{E}^{(n)}, H^1(\mathbb{R}^d))\) with
\[ \mathcal{E}^{(n)}(u, v) = \mathcal{G}(u, v) + \int_{\mathbb{R}^d} \nabla(u \nu)(x) \nabla \rho_n(x) \, dx - \int_{\mathbb{R}^d} u(x) \nu(x) \mu(dx) \]
\[ = \mathcal{G}(u, v) - \int_{\mathbb{R}^d} u(x) \nu(x) \Delta \rho_n(x) \, dx - \int_{\mathbb{R}^d} u(x) \nu(x) \mu(dx) \]
is the closed quadratic form corresponding to the semigroup \((T_t^{(n)})\); therefore, for any $\alpha > \beta$, any $f \in L^2(\mathbb{R}^d)$ and any $v \in H^1(\mathbb{R}^d)$,
\[ \mathcal{E}^{(n)}(R_\alpha^{(n)} f, v) = (f, v). \]

We know that $R_\alpha^{(n)} f$ converge to $R_\alpha f$ in $L^2(\mathbb{R}^d)$, so in order to prove (3.4) we need only to show that for $\alpha > \beta$, any $f \in L^2(\mathbb{R}^d)$ and any $v \in H^1(\mathbb{R}^d)$
\[ \lim_{n \to \infty} \mathcal{E}^{(n)}(R_\alpha^{(n)} f, v) = \mathcal{G}(R_\alpha f, v). \quad (3.5) \]
In the remainder of this proof, we assume that \( \alpha > \beta \), \( f \in L^2(\mathbb{R}^d) \) and \( v \in H^1(\mathbb{R}^d) \) are fixed. From Lemmas (3.1) and (3.2), we know that

\[
\lim_{n \to \infty} \mathcal{E}(R_\alpha^{(n)}f, v) = \mathcal{E}(R_\alpha f, v).
\]

Therefore, in order to prove (3.5), we need only to prove the following three identities:

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} R_\alpha^{(n)}f(x) \nabla \rho_n(x) \cdot \nabla v(x) \, dx = \int_{\mathbb{R}^d} R_\alpha f(x) \nabla \rho(x) \cdot \nabla v(x) \, dx,  
\]  

(3.6)

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} v(x) \nabla \rho_n(x) \cdot \nabla R_\alpha^{(n)}f(x) \, dx = \int_{\mathbb{R}^d} v(x) \nabla \rho(x) \cdot \nabla R_\alpha f(x) \, dx,  
\]  

(3.7)

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} v(x) R_\alpha^{(n)}f(x) \mu(dx) = \int_{\mathbb{R}^d} v(x) R_\alpha f(x) \mu(dx).  
\]  

(3.8)

Let us prove (3.6) first. From Lemma (2.1) and Lemma (3.1), we have

\[
\left| \int_{\mathbb{R}^d} R_\alpha^{(n)}f(x) \nabla(\rho_n - \rho)(x) \cdot \nabla v(x) \, dx \right| 
\leq \left( \int_{\mathbb{R}^d} |\nabla v|^2(x) \, dx \right)^{1/2} \left( \int_{\mathbb{R}^d} (R_\alpha^{(n)}f(x))^2 \frac{|\nabla(\rho_n - \rho)|^2}{\alpha} \, dx \right)^{1/2} 
\leq \left( \int_{\mathbb{R}^d} |\nabla v|^2(x) \, dx \right)^{1/2} M_d \left( \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\nabla(\rho_n - \rho)|^2(y)}{|x-y|^{d-2}} \, dy \right)^{1/2} \| R_\alpha^{(n)}f \|_{H^1}^2 
\leq C \left( \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\nabla(\rho_n - \rho)|^2(y)}{|x-y|^{d-2}} \, dy \right)^{1/2},
\]

thus by Proposition (1.3), we know that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} R_\alpha^{(n)}f(x) \nabla(\rho_n - \rho)(x) \cdot \nabla v(x) \, dx = 0.  
\]

(3.9)

Now from Theorem (2.3), we know that for any \( \varepsilon > 0 \), there exists a \( C(\varepsilon) > 0 \) such that

\[
\left| \int_{\mathbb{R}^d} (R_\alpha^{(n)}f - R_\alpha f)(x) \nabla \rho(x) \cdot \nabla v(x) \, dx \right| 
\leq \left( \int_{\mathbb{R}^d} |\nabla v|^2(x) \, dx \right)^{1/2} \left( \int_{\mathbb{R}^d} (R_\alpha^{(n)}f - R_\alpha f)^2(x) \, dx \right)^{1/2} 
\]
\[
\leq \left( \int_{\mathbb{R}^d} |\nabla v|^2 (x) \, dx \right)^{1/2} \\
\times \left( \varepsilon \int_{\mathbb{R}^d} |\nabla (R_z^{(n)} f - R_z f)|^2 (x) \, dx + C(\varepsilon) \int_{\mathbb{R}^d} (R_z^{(n)} f - R_z f)^2 (x) \, dx \right)^{1/2} \\
\leq M_1 \sqrt{\varepsilon} + \sqrt{C(\varepsilon)} \left( \int_{\mathbb{R}^d} |\nabla v|^2 (x) \, dx \right)^{1/2} \\
\times \left( \int_{\mathbb{R}^d} (R_z^{(n)} f - R_z f)^2 (x) \, dx \right)^{1/2},
\]

(3.10)

where

\[
M_1 = \left( \int_{\mathbb{R}^d} |\nabla v|^2 (x) \, dx \right)^{1/2} \cdot \sup_n \left\{ \left( \int_{\mathbb{R}^d} |\nabla (R_z^{(n)} f - R_z f)|^2 (x) \, dx \right)^{1/2} \right\}.
\]

Since \( R_z^{(n)} f \to R_z f \) in \( L^2(\mathbb{R}^d) \), we have

\[
\lim_{n \to \infty} \left| \int_{\mathbb{R}^d} (R_z^{(n)} f - R_z f)(x) \nabla \rho(x) \cdot \nabla v(x) \, dx \right| = 0.
\]

(3.11)

Using (3.9) and (3.11), we get (3.6).

Now let us go to the proof of (3.7). From Lemma (2.1) and Lemma (3.1), we have

\[
\left| \int_{\mathbb{R}^d} v(x) \nabla(\rho_n - \rho)(x) \cdot \nabla R_z^{(n)} f(x) \, dx \right|
\leq \left( \int_{\mathbb{R}^d} |\nabla R_z^{(n)} f|^2 (x) \, dx \right)^{1/2} \left( \int_{\mathbb{R}^d} v^2 (x) |\nabla(\rho_n - \rho)|^2 (x) \, dx \right)^{1/2}
\leq M_d \left( \int_{\mathbb{R}^d} |\nabla R_z^{(n)} f|^2 (x) \, dx \right)^{1/2}
\times \left( \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\nabla(\rho_n - \rho)|^2 (y)}{|x - y|^{d-2}} \, dy \right)^{1/2} \|v\|_{L^1}
\leq C \left( \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\nabla(\rho_n - \rho)|^2 (y)}{|x - y|^{d-2}} \, dy \right)^{1/2},
\]

thus by Proposition (1.3), we know that

\[
\lim_{n \to \infty} \left| \int_{\mathbb{R}^d} v(x) \nabla(\rho_n - \rho)(x) \cdot \nabla R_z^{(n)} f(x) \, dx \right| = 0.
\]

(3.12)
It follows from Theorem (2.3) that \( v |\nabla \rho | \) is in \( L^2(R^d) \); thus by Lemma (3.2), we know that

\[
\lim_{n \to \infty} \left| \int_{R^d} v(x) \nabla \rho(x) \cdot \nabla (R^{(n)}_zf - R_z f)(x) \, dx \right| = 0. \tag{3.13}
\]

Combining (3.12) and (3.13), we get (3.7).

Finally let us prove (3.8). From Theorem (2.3), we know that for any \( \varepsilon > 0 \) there exists a \( C(\varepsilon) > 0 \) such that

\[
\left| \int_{R^d} v(x) R^{(n)}_zf(x) \mu(dx) - \int_{R^d} v(x) R_z f(x) \mu(dx) \right| \\
\leq \left( \int_{R^d} v^2(x) \mu^*(x) \right)^{1/2} \left( \int_{R^d} (R^{(n)}_zf - R_z f)^2 (x) \mu^*(dx) \right)^{1/2} \\
\leq \left( \int_{R^d} v^2(x) \mu^*(x) \right)^{1/2} \\
x \left( \varepsilon^2 \int_{R^d} |\nabla (R^{(n)}_zf - R_z f)|^2 (x) \, dx \\
+ C(\varepsilon) \int_{R^d} (R^{(n)}_zf - R_z f)^2 (x) \, dx \right)^{1/2} \\
\leq M_2 \varepsilon + \sqrt{C(\varepsilon)} \left( \int_{R^d} v^2(x) \mu^*(x) \right)^{1/2} \\
x \left( \int_{R^d} (R^{(n)}_zf - R_z f)^2 (x) \, dx \right)^{1/2},
\]

where

\[
M_2 = \left( \int_{R^d} v^2(x) \mu^*(x) \right)^{1/2} \cdot \sup_n \left\{ \left( \int_{R^d} |\nabla (R^{(n)}_zf - R_z f)|^2 (x) \, dx \right)^{1/2} \right\},
\]

from which (3.8) follows.

Now the proof is complete.

In the light of the results of this article, we can now talk about solving the Dirichlet problem of the following equation:

\[
(\frac{1}{2}A + \Lambda \rho)v = 0.
\]

In fact, we can prove the following.

**Remark.** Let \( D \) be a bounded Lipschitz domain in \( R^d \). Suppose that the gauge function

\[
g(x) = E^x\{e^{A(x)\tau_D}\}
\]


is not identically infinite on $D$. If for any bounded measurable function $f$ on $\partial D$, we define

$$h(x) = E_x^x \{ f(X_{t_D}) \},$$
$$v(x) = E_x^x \{ e^{A(t_D)} f(X_{t_D}) \},$$
$$v_n(x) = E_x^x \{ e^{A_n(t_D)} f(X_{t_D}) \};$$

then

1. the sequence $v_n - h$ is bounded in $H^1_0(D)$ by some constant depending only on the $\rho$ and the $L^\infty$-norm of $v$;

2. $v$ is the unique continuous solution in the distributional sense to the Dirichlet problem of the equation

$$(\frac{1}{2} A + \Delta \rho) v = 0$$

with the boundary function $f$.

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