Intrinsic Ultracontractivity and Conditional Gauge for Symmetric Stable Processes

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It is shown in this paper that the conditional gauge theorem holds for symmetric \( \sigma \)-stable processes on bounded \( C^{1,1} \) domains in \( \mathbb{R}^n \) where \( 0 < \sigma < 2 \) and \( n \geq 2 \). Two of the major tools used to prove this conditional gauge theorem are logarithmic Sobolev inequality and intrinsic ultracontractivity.

1. INTRODUCTION

Suppose that \( W \) is a standard Brownian motion in \( \mathbb{R}^n \) with \( n \geq 3 \). If \( q \) is a function belonging to the Kato class and if \( D \) is a bounded domain in \( \mathbb{R}^n \), then a natural candidate for the solution of the Dirichlet problem

\[
\begin{cases}
\left( \frac{1}{2} A + q \right) g = 0, & \text{in } D, \\
g = 1, & \text{on } \partial D,
\end{cases}
\]

is

\[
g(x) = E^x \left[ \exp \left( \int_0^{\tau_D} q(W_t) \, dt \right) \right],
\]

where \( \tau_D \) is the first exit time of \( W \) from \( D \). In order to make sure when \( g \) is a solution to the problem above, however, one needs to study the finiteness of function \( g \). This is answered by the following theorem.

**Gauge Theorem.** The function \( g \) is either bounded or identically infinite on \( D \).
This theorem was first proved by Chung and Rao in [10] for bounded $q$ and later was generalized to more general $q$ by various authors. For the history of this theorem, we refer the reader to the recent book of Chung and Zhao [12]. The conditional gauge theorem, which is stated below, is related to the gauge theorem but is much deeper.

**Conditional Gauge Theorem.** If $D$ is a bounded $C^{1,1}$ domain and if the gauge function $g$ defined by (1.1) is finite for some $x \in D$, then

$$\sup_{x \in D, z \in \partial D} E_x^z \left[ \exp \left( \int_0^{\tau_D} q(W_t) \, dt \right) \right] < \infty,$$

where $E_x^z$ is the expectation with respect to the measure corresponding to the Brownian motion, starting from $x$, conditioned to exit $D$ from the point $z$.

The conditional gauge theorem was first proved by Falkner [17] for bounded $q$ and a class of domains including bounded $C^2$ domains. Extensions of this result to $q$ belonging to the Kato class and bounded $C^{1,1}$ domains were given by Zhao in [29] and [30]. The conditional gauge theorem has also been generalized by Cranston, Fabes and Zhao [13] to the case when $W$ is a diffusion process whose infinitesimal generator is a uniformly elliptic divergence form operator and $D$ is a bounded Lipschitz domain in $\mathbb{R}^n$.

Brownian motion is a special member in the family of symmetric stable processes. A symmetric $\alpha$-stable process $X$ on $\mathbb{R}^n$ is a Lévy process whose transition density $p(t, x-y)$ relative to Lebesgue measure is uniquely determined by its Fourier transform $\int_{\mathbb{R}^n} e^{i\langle x, y \rangle} p(t, x) \, dx = e^{-|x|^{1/\alpha}}$. Here $\alpha$ must be in the interval $(0, 2]$. Brownian motion is the symmetric 2-stable process, which has been intensively studied due to its central role in modern probability theory and its numerous important applications in other scientific areas including many other branches of mathematics. In the sequel, symmetric stable processes are referred to the case when $0 < \alpha < 2$, unless otherwise specified. In the last few years there has been an explosive growth in the study of physical and economic systems that can be successfully modeled with the use of stable processes. Stable processes are now widely used in physics, operation research, queuing theory, mathematical finance and risk estimation. See, for example, [21, 22] and the references therein. In order to make precise predictions about natural phenomena and to better cope with these widespread applications, there is a need to study the fine properties of symmetric stable processes, just as for the Brownian motion case. Although a lot is known about symmetric stable processes and their potential theory (see for example, [4, 21, 23] and the references therein), little is known until very recently about the counterparts to some of the deep results for Brownian motion, such as sharp estimates on Green
functions and Poisson kernels of bounded domains, and boundary Harnack principle for symmetric \( \alpha \)-stable process \( X \) in \( \mathbb{R}^n \) with \( n \geq 2 \) and \( 0 < \alpha < 2 \). Recently we obtained in \([9]\) sharp estimates for Green function \( G_D \) and Poisson kernel \( K_D \) of \( X \) in a bounded \( C^{1,1} \) domain \( D \).

In this paper, we continue our investigation started in \([9]\) about the fine properties for symmetric stable processes. More specifically we will show in this paper that conditional gauge theorem holds for symmetric stable processes in bounded \( C^{1,1} \) domains in \( \mathbb{R}^n \) with \( n \geq 2 \).

Recall that the Green function \( G_D \) and Poisson kernel \( K_D \) of \( X \) in \( D \) are determined by the following equations. For \( x \not\in D \),
\[
 E^x \left[ \int_{\tau_D} f(X_s) \, ds \right] = \int_D G_D(x, y) \, f(y) \, dy, \quad \text{for } f \geq 0 \text{ on } D,
\]
\[
 E^x \left[ \phi(X_{\tau_D}) \right] = \int_{D^c} K_D(x, z) \, \phi(z) \, dz, \quad \text{for } \phi \geq 0 \text{ on } D^c,
\]
where \( \tau_D = \inf \{ t > 0 : X_t \not\in D \} \). We proved in \([9]\) the following results.

**Theorem 1.1.** Let \( D \) be a bounded \( C^{1,1} \) domain in \( \mathbb{R}^n \) with \( n \geq 2 \). Then there is a constant \( c = c(D, \alpha) > 1 \) such that for any \( x, y \in D \),
\[
 c^{-1} \min \left\{ \frac{1}{|x - y|^{n-\alpha}}, \frac{\delta(x)^{\alpha/2} \delta(y)^{\alpha/2}}{|x - y|^{n}} \right\} \leq G_D(x, y) \leq c \min \left\{ \frac{1}{|x - y|^{n-\alpha}}, \frac{\delta(x)^{\alpha/2} \delta(y)^{\alpha/2}}{|x - y|^{n}} \right\}.
\]

**Theorem 1.2.** Let \( D \) be a bounded \( C^{1,1} \) domain in \( \mathbb{R}^n \) with \( n \geq 2 \). Then there is a constant \( c = c(D, \alpha) > 1 \) such that for any \( x \in D \) and any \( z \in D^c \),
\[
 \frac{\delta(x)^{\alpha/2}}{c \delta(z)^{\alpha/2} (1 + \delta(z))^{\alpha/2}} \frac{1}{|x - z|^{n}} \leq K_D(x, z) \leq \frac{c \delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2} (1 + \delta(z))^{\alpha/2}} \frac{1}{|x - z|^{n}}.
\]

The above estimates immediately imply the following 3G estimate.

**Theorem 1.3 (3G Theorem).** Suppose that \( D \) is a bounded \( C^{1,1} \) domain in \( \mathbb{R}^n \) with \( n \geq 2 \). Then there exists a constant \( c = c(D, \alpha) > 1 \) such that
\[
 \frac{G_D(x, y) G_D(y, w)}{G_D(x, w)} \leq \frac{c |x - w|^{n-\alpha}}{|x - y|^{n-\alpha} |y - w|^{n-\alpha}}, \quad x, y, w \in D, \quad (1.2)
\]
\[
 \frac{G_D(x, y) K_D(y, z)}{K_D(x, z)} \leq \frac{c |x - z|^{n-\alpha} |y - z|^{n-\alpha}}{|x - y|^{n-\alpha} |y - z|^{n-\alpha}}, \quad x, y \in D, \quad z \in D^c. \quad (1.3)
\]
Estimates like those in Theorems 1.1–1.3 are the keys in proving the conditional gauge theorem for Brownian motions (cf. [12]). However even with these estimates in hand, establishing the conditional gauge theorem for symmetric stable processes is far from easy. A look at all the known proofs of the conditional gauge theorem for Brownian motions and diffusion processes tells us that they all rely on the continuity of the sample paths of the process in an essential way. But a symmetric \( \alpha \)-stable process \( X \) with \( 0 < \alpha < 2 \) has discontinuous sample paths. It seems that a new approach is needed to deal with symmetric stable processes.

Besides using the estimates in Theorems 1.1–1.3, we establish in this paper the conditional gauge theorem for discontinuous symmetric stable processes on bounded \( C^{1,1} \) domains by showing that logarithmic Sobolev inequality and intrinsic ultracontractivity hold for symmetric stable processes. This approach of establishing conditional gauge theorem, as far as we know, is new even for Brownian motions.

The rest of this paper is organized as follows. The basic properties of symmetric stable processes and killed symmetric stable processes are studied in Section 2. The gauge theorem is discussed in Section 3. Part of the gauge theorem of Section 3 is covered by the general result of [11]. The reason that we have included this section is that we need the fact that the boundedness of the gauge function \( E[(\tau_D)] \) is equivalent to the first eigenvalue of the generator of the generalized Schrödinger operator \( L + q \) being negative (see Theorem 3.11 below), where \( L \) is the non-positive definite infinitesimal generator of the killed symmetric stable process. In Section 4, we show that the logarithmic Sobolev inequality holds for functions in the domain of the Dirichlet form associated with the killed symmetric \( \alpha \)-stable process on \( D \) and that the semigroup of the killed stable process on \( D \) is intrinsically ultracontractive. Under the assumption that the gauge function is finite in a bounded \( C^{1,1} \) domain \( D \), the Feynman–Kac semigroup is shown to be intrinsically ultracontractive and the Green function of the Feynman–Kac semigroup is controlled from above by the Green function of the killed stable semigroup in Section 5. The conditional gauge theorem is then shown to be true. Also contained in Section 5 are some important consequences of the conditional gauge theorem about the comparability of Green functions and Poisson kernels for \( L \) and \( L + q \) in bounded \( C^{1,1} \)-domains.

2. KILLED STABLE PROCESS

Throughout the remainder of this paper, we assume that \( n \geq 2 \) and \( 0 < \alpha < 2 \). Let \( X = \{ X_t \} \) be a symmetric \( \alpha \)-stable process in \( \mathbb{R}^n \). It is well known that the Dirichlet form \( (\mathcal{E}, \mathcal{F}) \) associated with \( X \) is given by
\[ e(u,v) = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+s}} \, dx \, dy \]

\[ \mathcal{F}^{R^n} = \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x-y|^{n+s}} \, dx \, dy < \infty \right\}. \]

As usual, we use \( \{ P_t \}_{t \geq 0} \) to denote the transition semigroup of \( X \) and \( G \) to denote the potential of \( \{ P_t \}_{t \geq 0} \); that is,

\[ Gf(x) = \int_0^\infty P_t f(x) \, dt. \]

**Theorem 2.1.** The semigroup \( \{ P_t \}_{t \geq 0} \) admits an integral kernel \( p(t, x, y) \) satisfying the following properties:

1. \( p(t, x, y) \) is strictly positive on \((0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n\);
2. \( p(t, x, y) \) is jointly continuous on \((0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n\);
3. for any \( t > 0 \) and any \( x, y \in \mathbb{R}^n \),
   \[ p(t, x, y) = p(t, y, x) = p(t, 0, y-x); \]
4. for any \( t > 0 \) and any \( x, y \in \mathbb{R}^n \),
   \[ p(t, x, y) = t^{-n}p(1, t^{-1}x, t^{-1}y); \]
5. there exists a constant \( c > 0 \) such that for any \( t > 0 \) and any \( x, y \in \mathbb{R}^n \), we have \( p(t, x, y) \leq ct^{-n/2} \);
6. there exists a constant \( c_0 = c_0(n, \alpha) > 0 \) depending only on \( n \) and \( \alpha \) such that
   \[ \lim_{|x-y| \to \infty} |x-y|^{n+s} p(1, x, y) = c_0. \]  \[ (2.1) \]

**Proof.** The assertions (1) to (5) are standard, the last assertion is the result of Theorem 2.1 in [6].

From the theorem above, one sees easily that \( P_t \) has both the Feller and strong Feller property. (See, for example, page 6 of [12] for the definitions of Feller property and strong Feller property.) For any set \( D \subset \mathbb{R}^n \), we are going to use \( \tau_D = \inf \{ t > 0 : X_t \not\in D \} \) to denote the first exit time of the symmetric \( \alpha \)-stable process \( X \) from \( D \).
Definition 2.1. (1) A boundary point $z$ of $D$ is said to be regular for $D$ if $P^\tau(\tau_D = 0) = 1$.

(2) $D$ is said to be regular if every boundary point of $D$ is regular for $D$.

Theorem 2.2. Let $z \in \partial D$. If there exists a cone $A$ with vertex $z$ such that $A \cap B(z, r) \subset D^c$ for some $r > 0$, then $z$ is regular for $D$. Here $B(z, r) = \{x \in \mathbb{R}^n : |x - z| < r\}$.

Proof. For $n \geq 1$, set $B_n = B(z, r/n)$ and $A_n = A \cap B_n^c$. Under $P^\tau(\cap_{n=1}^\infty \{X_{t_n} \in A_n\}) \subset \{\tau_D = 0\}$, hence

$$P^\tau(\tau_D = 0) \geq P^\tau\left(\bigcap_{n=1}^\infty \bigcup_{m=n}^\infty \{X_{t_m} \in A_n\}\right) \geq \limsup_n P^\tau(X_{t_n} \in A_n) = 0/\pi > 0,$$

where $\theta$ is the aperture of the cone $A$. The assertion of the theorem now follows from Blumenthal's zero-one law.

So any Lipschitz domain, and in particular, any $C^{1,1}$ domain, is regular. From now on, we assume that $D$ is a domain in $\mathbb{R}^n$. Adjoin an extra point $\partial$ to $D$ and set

$$X^\partial_t(\omega) = \begin{cases} X_t(\omega) & \text{if } t < \tau_D(\omega), \\ \partial & \text{if } t \geq \tau_D(\omega). \end{cases}$$

The process $X^\partial$ is called the symmetric $\alpha$-stable process killed upon leaving $D$, or simply the killed symmetric $\alpha$-stable process on $D$. It is well known (cf. [18]) that the Dirichlet form corresponding to the killed symmetric $\alpha$-stable process $X^\partial$ on $D$ is $(\mathcal{E}, \mathcal{F})$ where

$$\mathcal{F} = \{u \in \mathcal{F}^{\mathbb{R}^n} : u = 0 \text{ quasi everywhere on } D^c\}.$$

For $t > 0$, $x \in D$, and $f \in L^\infty(D)$, set

$$P^\partial_t f(x) = E^\partial[ f(X_t); t < \tau_D].$$

Repeating the arguments of Section 2.1 of [12], we get the following result.

Theorem 2.3. For any domain $D \subset \mathbb{R}^n$ we have $P^\partial_t f \in C_0(D)$ for $t > 0$ and $f \in L^\infty(D)$. Moreover, if $D$ is regular, then $P^\partial_t f \in C_0(D)$ for $f \in C_0(D)$. In the latter case, $X^\partial$ on $D$ has both the Feller and the strong Feller property.
Here and hereafter $C_b(D)$ is the space of bounded continuous functions in $D$ and $C_0(D)$ is the space of continuous functions in $D$ that vanish on $\partial D$.

For $t > 0$, $x, y \in \mathbb{R}^n$, let

$$r^D(t, x, y) = E^x[p(t - \tau_D, X_{\tau_D}; \tau_D < t)]$$

and

$$p^D(t, x, y) = p(t, x, y) - r^D(t, x, y).$$

Note that by the right continuity of the sample paths of $X$, we have $p^D(t, x, y) = 0$ for $x \in \mathbb{R}^n \setminus D$.

**Theorem 2.4.** Let $D$ be a domain in $\mathbb{R}^n$. Then for any $t > 0$, $x \in \mathbb{R}^n$ and nonnegative Borel measurable function $f$ on $\mathbb{R}^n$,

$$p^D f(x) = \int_{\mathbb{R}^n} p^D(t, x, y) f(y) \, dy.$$ 

The function $p^D(t, \cdot, \cdot)$ is symmetric on $\mathbb{R}^n \times \mathbb{R}^n$ and strictly positive on $D \times D$. As a function of $(t, x, y)$, $p^D$ is continuous on $(0, \infty) \times (\mathbb{R}^n \setminus \partial D) \times (\mathbb{R}^n \setminus \partial D)$.

For any $t, s > 0$, $x, y \in \mathbb{R}^n$, we have

$$p^D(t + s, x, y) = \int_{\mathbb{R}^n} p^D(t, x, z) p^D(s, z, y) \, dz.$$

For any $t > 0$, $y \in D$ and a regular point $z \in \partial D$, we have

$$\lim_{D \times x \to z} p^D(t, x, y) = 0.$$

**Proof.** All the assertions of this theorem except the strict positivity of $p^D$ can be proven by using an argument similar to that of Theorem 2.4 of [12] or that of Theorem 2.4.3 of [24]. So we are only going to prove the strict positivity of $p^D(t, \cdot, \cdot)$ on $D \times D$.

Let $a$ be an arbitrary fixed point in $D$ and $r > 0$ be such that $r < d(a, \partial D)/6$. For any $x \in B(a, r)$, $y \in B(a, 3r) \setminus B(a, 2r)$, we have $4r > |x - y| > r$ and $|X_0 - y| > 3r$. Using (2.1) we know that there is $t_0 > 0$ such that when $0 < t < t_0$, 

$$p(t, x, y) = t^{-n} p(1, t^{-1/2}, t^{-1/2} x, t^{-1/2} y) \geq \frac{c_0}{2} \frac{t}{|x - y|^{n+\epsilon}} \geq \frac{c_0}{2} \frac{t}{(4r)^{n+\epsilon}},$$
and when $0 < t \leq t_0$ with $t - \tau_D > 0$

$$p(t - \tau_D, X_{\tau_D}, y) = (t - \tau_D)^{-n+} p(1, (t - \tau_D)^{-1/\gamma} X_{\tau_D}, (t - \tau_D)^{-1/\gamma} y)$$

$$\leq 2c_0 (t - \tau_D)^{-n+} \frac{1}{(t - \tau_D)^{-1/\gamma} (X_{\tau_D} - y)^{n+}}$$

$$= 2c_0 \frac{t - \tau_D}{|X_{\tau_D} - y|^{n+}} \leq 2c_0 \frac{t}{(3r)^{n+}}.$$  

Therefore when $0 < t \leq t_0$,

$$\frac{r_D(t, x, y)}{p(t, x, y)} \leq 4 \left(\frac{4}{3}\right)^{n+} t^{n+} \leq 4 \left(\frac{4}{3}\right)^{n+} \frac{p_D(\tau_D, \tau_D) < t}{p_D(\tau_D, \tau_D) < t}. $$

Hence

$$\lim_{i \to 0} \frac{r_D(t, x, y)}{p(t, x, y)} = 0$$

uniformly for $(x, y) \in B(a, r) \times (B(a, 3r) \B (a, 2r));$ or equivalently

$$\lim_{i \to 0} \frac{p_D(t, x, y)}{p(t, x, y)} = 1$$  \hspace{1cm} (2.2)

uniformly for $(x, y) \in B(a, r) \times (B(a, 3r) \B (a, 2r)).$ Therefore there exists $t_1 > 0$ such that for any $0 < t \leq t_1$ and $(x, y) \in B(a, r) \times (B(a, 3r) \B (a, 2r)),

$p_D(t, x, y) > 0.$ Since

$$p_D(t, x, z) = \int_{\mathbb{R}^n} p_D(t/2, x, y) p_D(t/2, y, z) \, dy$$

$$\geq \int_{B(a, 3r) \B (a, 2r)} p_D(t/2, x, y) p_D(t/2, y, z) \, dy$$

we have $p_D(t, x, z) > 0$ for any $(x, z) \in B(a, r) \times B(a, r)$ whenever $0 < t \leq t_1$.

Now we are in a position to prove the strict positivity of $p_D$ on $D \times D$. For $x \in D$, we need to show that

$$p_D(t, x, y) > 0 \hspace{1cm} \forall t > 0, \hspace{1cm} \forall y \in D.$$  

To this end, set

$$U = \{ y \in D : p_D(t, x, y) > 0 \hspace{1cm} \text{for all} \hspace{1cm} t > 0 \}.$$
From the last paragraph we know that there exist $t_0 > 0$ and $r > 0$ such that $p^0(s, z, y) > 0$ for all $0 < s \leq t_0$ and $y, z \in B(x, r)$. It then follows from the semigroup property of $p^0$ that $p^0(t, x, y) > 0$ for all $t > 0$ and $y \in B(x, r)$. Thus $U$ is nonempty.

We next show that $U = D$. Since $D$ is connected, it suffices to show that $\overline{U} \cap D$ is contained in the interior of $U$. Suppose that $y_0 \in \overline{U} \cap D$. By a similar reasoning as in the last paragraph, we know that there exists $r_1 > 0$ such that $p^0(t, z, y) > 0$ for all $t > 0$ and $z, y \in B(y_0, r_1)$. Take $y_1 \in U \cap B(y_0, r_1)$. Since $p^0(t/2, x, y_1) > 0$, so by continuity $p^0(t/2, z, y) > 0$ for all $z$ in a subset of $B(y_0, r_1)$ having positive measure. Thus

$$p^0(t, x, y) \geq \int_{B(y_0, r_1)} p^0(t/2, x, z) p^0(t/2, z, y) \, dz > 0$$

for all $t > 0$ and $y \in B(y_0, r_1)$. Therefore $B(y_0, r_1) \subseteq U$.

For any domain $D$ and $1 \leq p < \infty$, $L^p(D)$ will be called an "appropriate space" for $D$. If in addition $D$ is regular, then $C_0(D)$ will also be called an appropriate space for $D$.

**Theorem 2.5.** Let $D$ be a domain in $\mathbb{R}^n$. Then for each appropriate space $S$ for $D$, $\{p^0(t, x) : t \geq 0\}$ forms a strongly continuous semigroup in $S$. If, in addition, $|D| < \infty$, then for each $t > 0$, $P^0_t$ is a bounded operator from $S_1$ to $S_2$ for any two appropriate spaces $S_1$ and $S_2$ for $D$. Furthermore, $P^0_t$ is a compact operator and has the same eigenvalues $\{e^{\lambda_k t} : k = 1, 2, \ldots\}$ with $\lambda_k < 0$ in all the appropriate spaces for $D$. Suppose in addition that $D$ is regular. Then for each $t > 0$, $P^0_t$ is a bounded operator from $L^p(D)$ into $C_0(D)$.

**Proof.** The proof is the same as that of Theorem 2.7 of [12] with some minor and obvious modifications.

3. GAUGE THEOREM

Recall that $0 < n < 2$ and $n \geq 2$.

**Definition 3.1.** A Borel measurable function $q$ on $\mathbb{R}^n$ is said to be in the Kato class $K_{n, x}$ if

$$\lim_{r \downarrow 0} \sup_{x \in \mathbb{R}^n} \int_{|x - y| < r} \frac{|q(y)|}{|x - y|^{n-x}} \, dy = 0.$$ 

We are going to need the following result on the decomposition of Kato class functions later on.
LEMMA 3.1. Let \( q \) have compact support. Then \( q \in K_{n, \infty} \) if and only if, for any \( \varepsilon > 0 \), there is a function \( q_\varepsilon \) such that \( q - q_\varepsilon \) is bounded and

\[
\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|q_\varepsilon(y)|}{|x-y|^{n+2}} \, dy \leq \varepsilon.
\]

Proof. The proof is the same as that of Theorem 4.16 in [1].

The following probabilistic characterization of Kato class functions is due to Zhao [31].

THEOREM 3.2. A Borel measurable function \( q \) is in the Kato class \( K_{n, \infty} \) if and only if

\[
\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^n} \left[ \int_0^t |q(X_s)| \, ds \right] = 0.
\]

In what follows, we are going to assume that \( q \) is an arbitrary, but fixed function in \( K_{n, \infty} \). For \( t > 0 \), let

\[
eq(t) = \exp \left( \int_0^t q(X_s) \, ds \right).
\]

For a domain \( D \) in \( \mathbb{R}^n \), define

\[
T_t f(x) = E\left[ e^t f(X(t)); t < \tau_D \right], \quad x \in D.
\]

THEOREM 3.3. Let \( D \) be a domain in \( \mathbb{R}^n \), and \( \{T_t\} \) be defined as above. Then \( \{T_t\} \) is a strongly continuous semigroup in each appropriate space for \( D \). Each \( T_t \) is a bounded operator from \( L^p(D) \), \( 1 \leq p \leq \infty \), to \( L^\infty(D) \) and to itself, and there exist \( c_1, c_2 > 0 \) such that

\[
\|T_t\|_p \leq \|T_t\|_\infty \leq e^{c_1 + c_2 t}, \quad 0 \leq t < \infty. \tag{3.1}
\]

For each \( t > 0 \), \( T_t \) has the strong Feller property and possesses a symmetric density \( u(t, \cdot, \cdot) \in C_0(D \times D) \) such that for any \( f \in L^p(D) \), \( 1 \leq p \leq \infty \),

\[
T_t f(x) = \int_D u(t, x, y) f(y) \, dy, \quad x \in D.
\]

\( T_t \) maps \( L^1(D) \) into \( C_0(D) \).

Suppose in addition that \( |D| < \infty \) and that \( D \) is regular. Then for each \( t > 0 \), \( u(t, \cdot, \cdot) \in C_0(D \times D) \); \( T_t \) is a bounded operator from \( L^\infty(D) \) into \( C_0(D) \). It is a compact operator in all appropriate spaces, and has the same
eigenvalues and eigenfunctions in all of them. All the eigenfunctions belong to \( C_0(D) \).

Proof. The proof is the same as that of Theorem 3.17 in [12].

**Theorem 3.4.** There exist constants \( c_1, c_2 > 0 \) such that for any \( t > 0 \), \( x, y \in D \),

\[
    u(t, x, y) \leq c_1 e^{c_2 t} t^{-n/\alpha}.
\]

Proof. In the proof of this theorem, \( c_1 \) and \( c_2 \) will denote positive constants whose value may change from line to line. By (3.1) and the fact that there exists \( c > 0 \) such that \( p(t, x, y) \leq ct^{-n/\alpha} \) for \( t > 0 \), we have

\[
    |T_t f(x)|^2 \leq E^\star \left[ e_{c_2}(t); t < \tau_D \right] E^\star \left[ t < \tau_D; f(X_t)^2 \right] \leq c_1 e^{c_2 t} t^{-n/\alpha} \| f \|^2_2.
\]

Thus

\[
    \| T_t \|_{L^\infty} \leq c_1 e^{c_2 t} t^{-n/\alpha} \tag{3.2}
\]

and consequently

\[
    \| T_t \|_{L^1} \leq \| T_{c_2} \|_{L^2} \| T_{c_2} \|_{L^\infty} \leq \| T_{c_2} \|^2_{L^\infty} \leq c_1 e^{c_2 t} t^{-n/\alpha}.
\]

Therefore, for any Borel subset \( B \) of \( D \), we have

\[
    T_{c_2} 1_B(x) \leq c_1 e^{c_2 t} t^{-n/\alpha} |B|.
\]

Now the conclusion of the theorem follows immediately.

**Theorem 3.5.** \( u(t, x, y) > 0 \) on \((0, \infty) \times D \times D\).

Proof. For \( x \in D \) and a Borel measurable function \( f \) which is positive on a subset of \( D \) with positive measure, we have

\[
    \int_D u(t, x, y) f(y) dy = E^\star \left[ e_{c_2}(t); t < \tau_D \right] f(X(t)) \geq 0
\]

which shows that, for any \( x \in D \),

\[
    u(t, x, y) > 0, \quad \text{for almost every } y \in D.
\]

By symmetry, we have for any \( y \in D \),

\[
    u(t, x, y) > 0, \quad \text{for almost every } x \in D.
\]

The theorem now follows from the semigroup property.
Theorem 3.6. Suppose that $\phi$ is an eigenfunction of $T_t$ corresponding to the eigenvalue $e^{i\lambda}$. If $\phi$ is nonnegative, then $\phi$ is strictly positive on $D$.

Proof. If $\phi(x) = 0$ for some $x \in D$, then we have for any $t > 0$,

$$0 = e^{i\lambda} \phi(x) = T_t \phi(x) = \int_D u(t, x, y) \phi(y) \, dy.$$ 

Since $u(t, x, \cdot)$ is strictly positive on $D$, $\phi(y) = 0$ for all $y \in D$, which is a contradiction.

In the remainder of this section we are going to assume that $D$ is a regular domain in $\mathbb{R}^n$ with $|D| < \infty$.

Let $\{e^{i\lambda_k} : k = 1, 2, \ldots\}$ be all the eigenvalues of $T_t$ written in decreasing order, each repeated according its multiplicity. Then $\lambda_k \downarrow -\infty$ and the corresponding eigenfunctions $\{\phi_k\}$ can be so chosen that they form an orthonormal basis of $L^2(D)$. We also know that the eigenspace corresponding to the first eigenvalue is one dimensional and $\phi_1$ can be chosen to be nonnegative. In fact, $\phi_1$ can be chosen to be strictly positive on $D$.

Let $f \in L^2(D)$. Then there exist $a_i, i = 1, 2, \ldots$ such that for almost every $x \in D$,

$$f(x) = \sum_{i=1}^{\infty} a_i \phi_i(x).$$

Hence, for any $t > 0$ and every $x \in D$,

$$T_t f(x) = \sum_{i=1}^{\infty} a_i e^{i\lambda_i t} \phi_i(x). \quad (3.3)$$

In fact we have:

Theorem 3.7. If $f = \sum_{i=1}^{\infty} a_i \phi_i \in L^2(D)$, then for any $t > 0$

$$\lim_{n \to \infty} \sup_{x \in D} \left| T_t f(x) - \sum_{i=1}^{n} a_i e^{i\lambda_i t} \phi_i(x) \right| = 0. \quad (3.4)$$

Proof. (3.4) follows easily from (3.2).

From the above result, we can easily get the following result on the density function $u(t, x, \cdot)$ for the semigroup $T_t$. 
Theorem 3.8. For any \( t > 0 \) and any \( x, y \in D \),

\[
u(t, x, y) = \sum_{i=1}^{\infty} e^{\lambda_i t} \phi_i(x) \phi_i(y).
\]

Proof. The proof is straightforward and is omitted.

Theorem 3.9. (1) Let \( f \in L^2(D) \) and \( \lambda \) be a real number. Then for each \( x \in D \), the limit

\[
S_\lambda f(x) := \lim_{t \to \infty} e^{-\lambda t} T_t f(x)
\]

always exists in \([-\infty, \infty]\). If \( |S_\lambda f(x)| < \infty \) on a dense subset of \( D \), then

\[
\lim_{t \to \infty} \sup_{x \in D} e^{-\lambda t} |T_t f(x)|
\]

exists and is finite.

(2) Let \( f \in L^2(D) \) with \( \|f\|_2 > 0 \). Define

\[
\gamma = \inf \{ \lambda \in \mathbb{R} : \lim_{t \to \infty} \sup_{x \in D} e^{-\lambda t} |T_t f(x)| < \infty \}.
\]

Then \( S_\gamma f \) is an eigenfunction of \( T_t \) corresponding to the eigenvalue \( e^{\gamma t} \).

Proof. Let \( f \neq 0 \) in \( L^2(D) \). Then \( f \) can be expressed as \( \sum_{i=1}^{\infty} a_i \phi_i \). So there exists at least one \( j \geq 1 \) such that \( \sum_{i=1}^{\infty} a_i \phi_i \neq 0 \). Let

\[
k = \min \left\{ j \geq 1 : \sum_{i : \lambda_i = \lambda_j} a_i \phi_i \neq 0 \right\}.
\]

Then by Theorem 3.7 we have for any \( 0 < \varepsilon < t \),

\[
e^{-\lambda t} T_t f(x) = e^{\lambda_k - \lambda_j t} \sum_{i : \lambda_i = \lambda_k} e^{\lambda_i t} a_i \phi_i(x) + \sum_{i : \lambda_i < \lambda_j} e^{\lambda_i t} a_i \phi_i(x) \times \left( \sum_{i : \lambda_i = \lambda_k} e^{\lambda_i \varepsilon} e^{\lambda_i \varepsilon} + \sum_{i : \lambda_i < \lambda_k} e^{\lambda_i \varepsilon} \right) e^{-\lambda_j \varepsilon} \phi_j(x) \right) \times \left( \sum_{i : \lambda_i = \lambda_k} e^{\lambda_i \varepsilon} e^{\lambda_i \varepsilon} + \sum_{i : \lambda_i < \lambda_k} e^{\lambda_i \varepsilon} \right) e^{\lambda_i \varepsilon} \phi_i(x) \right).
\]

Define

\[
\eta = \min \{ \lambda_k - \lambda_j : \lambda_j < \lambda_k \},
\]

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then $\eta > 0$. Since by Theorem 3.4
\[
\left( \sum_{i=1}^{\infty} e^{t^i} |a_i \phi_i(x)| \right)^2 \leq \left( \sum_{i=1}^{\infty} a_i \right)^2 \left( \sum_{i=1}^{\infty} |e^{t^i} \phi_i(x)|^2 \right) \\
= \|f\|_*^2 \|u(\kappa, x, \cdot)|^2 \\
= \|f\|_*^2 \|u(2\kappa, x, x) \\
\leq \|f\|_*^2 c_1 e^{2\kappa(2\kappa)^{-m_*}},
\]
therefore
\[
\limsup_{t\to\infty} \sum_{i=1}^{\infty} e^{t_i^{-1} - \lambda_i^{-1}(-t_i^{-1})} |e^{t^i} a_i \phi_i(x)| \\
\leq \limsup_{t\to\infty} e^{-\eta(t_i^{-1})} \sup_{x \in D} \sum_{i=1}^{\infty} e^{t^i} |a_i \phi_i(x)| = 0. \tag{3.7}
\]
From (3.6) and (3.7) we get that
(a) If $\lambda_k - \lambda > 0$, then for any $x \in D$,
\[
\lim_{t\to\infty} e^{-\lambda_k T_t f(x)} = \begin{cases} 
\infty, & \text{if } \sum_{j=1}^{\lambda_k} a_j \phi_j(x) > 0; \\
-\infty, & \text{if } \sum_{j=1}^{\lambda_k} a_j \phi_j(x) < 0.
\end{cases}
\]
(b) If $\lambda_k - \lambda = 0$, then for any $x \in D$ we have
\[
\lim_{t\to\infty} e^{-\lambda_k T_t f(x)} = \sum_{j=1}^{\lambda_k} a_j \phi_j(x).
\]
(c) If $\lambda_k - \lambda < 0$, then for any $x \in D$, we have
\[
\lim_{t\to\infty} e^{-\lambda_k T_t f(x)} = 0.
\]
Thus the limit in (3.5) always exists in $[-\infty, \infty]$. The rest of the assertion just follows from (a)–(c).

**Lemma 3.10.** If $\phi \in L^2(D)$ is strictly positive on $D$, then the following assertions are equivalent.

1. There exists $x \in D$ such that
\[
\lim_{n\to\infty} T_{\lambda_1} \phi(x) = 0.
\]
2. $\lambda_1 < 0$. 
Proof. Write \( \phi = \sum_{i=1}^{\infty} a_i \phi_i \). Then by Theorem 3.7 we know that for any \( t > 0 \) and any \( x \in D \),
\[
T_t \phi(x) = \sum_{i=1}^{\infty} e^{t \lambda_i} a_i \phi_i(x).
\]
The first eigenfunction \( \phi_1 \) is strictly positive on \( D \). Therefore if \( \phi \) is strictly positive on \( D \), then \( a_1 = \int_D \phi_1(x) \phi(x) \, dx > 0 \). Thus by the proof of Theorem 3.9, the assertions (1) and (2) are equivalent. 

The following is the gauge theorem.

**Theorem 3.11.** Let \( D \) be a regular domain with \(|D| < \infty\). The following assertions are equivalent:

1. \( E^*[e_{\gamma}(\tau_D)] < \infty \) for some \( x \in D \);
2. the function \( g(x) = E^*[e_{\gamma}(\tau_D)] \) is bounded on \( D \);
3. there exists a nonnegative function \( \tilde{q} \in K_{n,s} \) which is strictly positive on a subset of \( D \) having positive Lebesgue measure such that
\[
E^* \left[ \int_0^{\tau_D} e_{\gamma}(X) \tilde{q}(X) \, dt \right] < \infty
\]
for some \( x \in D \);
4. for all nonnegative function \( \tilde{q} \in K_{n,s} \), the function
\[
x \mapsto E^* \left[ \int_0^{\tau_D} e_{\gamma}(t) \tilde{q}(X) \, dt \right]
\]
is bounded on \( D \);
5. \( \lambda_1 < 0 \).

**Proof.** (1) \( \Rightarrow \) (5): For \( x \in D \), define
\[
\phi(x) = E^*[e_{\gamma}(\tau_D); \tau_D \leq 1],
\]
which is strictly positive on \( D \). By Khas'minskii's lemma and Theorem 3.2, there is a positive integer \( m > 1 \) such that \( \sup_{x} \psi(x) < \infty \), where \( \psi(x) = E^*[e_{\gamma}(\tau_D); \tau_D \leq 1/m] \). Therefore \( \phi = \sum_{k=0}^{m-1} T_{km} \psi(x) \) is bounded on \( D \) and so \( \phi \in L^2(D) \). By the Markov property of \( X \),
\[
E^*[e_{\gamma}(\tau_D)] = \phi(x) + \sum_{n=1}^{\infty} T_n \phi(x) \tag{3.8}
\]
for every \( x \in D \). Since \( E^x [e_x(\tau_D)] < \infty \) for some \( x \in D \) by (1), we know that
\[
\lim_{n \to \infty} T_n \phi(x) = 0.
\]

Therefore by Lemma 3.10, \( \lambda_1 < 0 \).

(5) \( \Rightarrow \) (2): Let \( \beta = \lambda_1 / 2 < 0 \). By the proof of Theorem 3.9 we get
\[
\lim_{t \to \infty} \sup_{x \in D} e^{-\beta t} |T_t \phi(x)| < \infty.
\]

Therefore by (3.8),
\[
\sup_{x \in D} E^x [e_x(\tau_D)] \leq \|\phi\|_\infty + \sum_{n=1}^\infty e^{\beta n} \sup_{x \in D} e^{-\beta n} |T_n \phi(x)| < \infty.
\]

(2) \( \Rightarrow \) (1) and (4) \( \Rightarrow \) (3) are trivial.

(3) \( \Rightarrow \) (5) \( \Rightarrow \) (4): For a nonnegative function \( \tilde{q} \in K_{n, \pi} \), let
\[
\psi(x) = E^x \left[ \int_0^{\tau_{D^\pi}} e_x(t) \tilde{q}(X_t) \, dt \right].
\]

Since
\[
\psi(x) \leq (E^x [e_{2\pi}(\tau_D \land 1)])^{1/2} \left( E^x \left[ \left( \int_0^{\tau_{D^\pi}} \tilde{q}(X_t) \, dt \right)^2 \right] \right)^{1/2},
\]

thus by Khas’minskii’s inequality and Theorem 3.2 we see that \( \psi \) is bounded on \( D \). If \( \tilde{q} \) is strictly positive on a subset of \( D \) with positive Lebesgue measure, then \( \psi \) is strictly positive on \( D \). Thus to complete the proof we use the same arguments as in the previous part of the proof.

4. INTRINSIC ULTRACONTRACTIVITY

First we are going to recall the definition of intrinsic ultracontractivity which is due to Davies and Simon [15]. Suppose that \( H \) is a semibounded self-adjoint operator on \( L^2(D) \) with \( D \) being a domain in \( \mathbb{R}^n \) and that \( e^{Ht} \) is an irreducible positivity-preserving semigroup with integral kernel \( a(t, x, y) \).

We assume that the top of the spectrum \( \lambda_1 \) of \( H \) is an eigenvalue. In this case, \( \lambda_1 \) has multiplicity one and the corresponding eigenfunction \( \phi_1 \), normalized by \( \|\phi_1\|_2 = 1 \), is positive almost everywhere on \( D \). \( \phi_1 \) is called the ground state of \( H \).

We now define the unitary operator \( U \) from \( L^2(D, \phi_1^2(x) \, dx) \) to \( L^2(D) \) by \( Uf = \phi_1 f \) and define \( \widehat{H} \) on \( L^2(D, \phi_1^2(x) \, dx) \) by
\[
\widehat{H} = U^{-1} (H - \lambda_1) U.
\]
Then $e^{B_t}$ is an irreducible symmetric Markov semigroup on $L^2(D, \phi^2(x) \, dx)$ whose integral kernel with respect to the measure $\phi^2(x) \, dx$ is given by

$$e^{-\lambda t} \frac{a(t, x, y)}{\phi(x) \phi(y)}.$$

**Definition 4.1.** $H$ is said to be ultracontractive if $e^{B_t}$ is a bounded operator from $L^2(D)$ to $L^\infty(D)$ for all $t > 0$. $H$ is said to be intrinsically ultracontractive if $H$ is ultracontractive; that is, $e^{B_t}$ is a bounded operator from $L^2(D, \phi^2(x) \, dx)$ to $L^\infty(D, \phi^2(x) \, dx)$ for all $t > 0$.

Ultracontractivity is connected to logarithmic Sobolev inequalities. The connection between logarithmic Sobolev inequalities and $L^p$ to $L^q$ bounds of semigroups was first demonstrated by L. Gross [19] in 1975. E. Davies and B. Simon [1] adopted Gross’s approach to allow $q = \infty$ and therefore established the connection between logarithmic Sobolev inequalities and ultracontractivity. (For an updated survey on the subject of logarithmic Sobolev inequalities and contractivity properties of semigroups, see [2, 20].)

In [3], R. Bañuelos proved the intrinsic ultracontractivity for Schrödinger operators on uniformly Hölder domains of order $\alpha \in (0, 2)$ using logarithmic Sobolev inequality characterization. We will use the same strategy in this section; that is, establishing intrinsic ultracontractivity through logarithmic Sobolev inequalities.

In the sequel, unless otherwise specified, $D$ is a bounded $C^{1,1}$ domain in $\mathbb{R}^n$ where $n \geq 2$. Let $X^\alpha$ be the symmetric $\alpha$-stable process killed upon leaving $D$ for some fixed $\alpha \in (0, 2)$, and $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form of $X^\alpha$ on $L^2(D, dx)$. The non-positive definite infinitesimal generator of $X^\alpha$ is denoted by $L$. In this section we will show that $L$ is intrinsically ultracontractive, by establishing logarithmic Sobolev inequalities through logarithmic Sobolev inequalities.

**Theorem 4.1.** The logarithmic Sobolev inequality holds for functions in $(\mathcal{E}, \mathcal{F})$. That is, for any $\eta > 0$ and $f \in \mathcal{F} \cap L^\infty(D, dx)$, we have

$$\int_D f^2 \log |f| \, dx \leq \eta \mathcal{E}(f, f) + \beta(\eta) \|f\|_2^2 + \|f\|_2^2 \log \|f\|_2,$$

with

$$\beta(\eta) = -\frac{\eta}{2\alpha} \log \eta + c$$

for some constant $c > 0$.

**Proof.** From Theorem 2.1(5) we know that there exists $c > 0$ such that $p(t, x, y) \leq ct^{-n\alpha}$. Thus for any $f \in L^2(D), \ldots$
\[ |P^D_t f(x)| = \left| \int_D p^D(t, x, y) f(y) \, dy \right| \leq \int_D |p(t, x, y)| f(y) \, dy \]
\[ \leq \sqrt{\int_{\mathbb{R}^n} p(t, x, y)^2 \, dy} \| f \|_2 = \sqrt{p(2t, x, x)} \| f \|_2 \]
\[ \leq \sqrt{c(2t)^{-n}} \| f \|_2. \]

Hence \( \| P^D_t \|_{\infty, 2} \leq c t^{-(n/2)} \). Take \( e^{M(t)} = c t^{-(n/2)} \), that is, \( M(t) = -n(2t) \log t + \log c \). Then it follows from Theorem 2.2.4 of [14] that for any \( \eta > 0 \) and \( f \in \mathcal{F} \cap L^\infty(D, dx) \),
\[ \int_D f^2 \log |f| \, dx \leq \eta \mathcal{E}(f, f) + \beta(\eta) \| f \|_2^2 + \| f \|_2^2 \log \| f \|_2 \]
with
\[ \beta(\eta) = M \left( \frac{\eta}{4} \right) + 2 = -\frac{n}{2x} \log \eta + c. \]

**Theorem 4.2.** Let \( \phi_1 \) be the ground state of \( L \). Then there exists \( c > 1 \) such that
\[ c^{-1} \delta(x)^{n^2} \leq \phi_1(x) \leq c\delta(x)^{n^2}, \]
where \( \delta(x) \) is the Euclidean distance between \( x \) and \( \partial D \).

**Proof.** We know that the first eigenvalue \( \lambda_1 \) of \( L \) is negative and that \( \phi_1 \in C_0(D) \) is strictly positive in \( D \). Thus by Theorem 1.1
\[ \phi_1(x) = -\lambda_1^{-1} G_D \phi_1(x) = -\lambda_1^{-1} \int_D G_D(x, y) \phi_1(y) \, dy \]
\[ \geq -\lambda_1^{-1} \delta(x)^{n^2} c \int_D \left( \frac{\delta(y)^{n^2}}{|x-y|} \wedge \frac{1}{\delta(x)^{n^2}|x-y|^{n-2}} \right) \phi_1(y) \, dy \geq c\delta(x)^{n^2}. \]

Note that \( \phi_1 = e^{-\lambda_1 T} \phi_1 \) is bounded by Theorem 3.3. Therefore by Theorem 1.1 of [9]
\[ \phi_1(x) = -\lambda_1^{-1} \int_D G_D(x, y) \phi_1(y) \, dy \leq c \int_D \frac{\delta(y)^{n^2}}{|x-y|^{n-2}} \phi_1(y) \, dy \]
\[ \leq c \delta(x)^{n^2}. \]
LEMMA 4.3. Suppose that $D$ is a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$ and let $\delta(x) = d(x, \partial D)$ be the distance from $x$ to the boundary $\partial D$ of $D$. Then $\log \delta \in L^p(D)$ for any $p > 0$.

Proof. Since $D$ is a bounded Lipschitz domain, there exist a positive constant $r \in (0, \frac{1}{2})$ and finitely many points $\{x_1, ..., x_k \} \subset \partial D$ such that $\partial D \subset \bigcup_{i=1}^k B(x_i, r)$ and that for each $1 \leq i \leq k$, $B(x_i, 4r) \cap D$ is the region above the graph of a Lipschitz function in $B(x_i, 4r)$. That is, for each $i \in \{1, 2, ..., k\}$, there exits a coordinate system $(\xi^i, \zeta^i)$, where $\xi^i \in \mathbb{R}$ and $\zeta^i \in \mathbb{R}^{n-1}$, with origin sitting at $x_i$ and a Lipschitz function $f_i$ defined on $\mathbb{R}^{n-1}$ such that $B(x_i, 4r) \cap D = B(x_i, 4r) \cap \{ \xi = (\xi^i, \zeta^i) : \zeta^i > f_i(\zeta^i) \}$ and $\partial D \cap B(x_i, 4r) = \{ (\xi, \zeta^i) : \xi = f_i(\zeta^i) \} \cap B(x_i, 4r)$. For each $\xi \in B(x_i, r) \cap D$, let $y \in \partial D$ be such that $|\xi - y| = \delta(\xi)$. Clearly $y \in B(x_i, 4r)$.

Let $\hat{y} = (f_i(\zeta^i), \zeta^i) \in \partial D \cap B(x_i, r)$. Denote the Lipschitz constant of $f_i$ by $s_i$. Let $\Gamma_M(\hat{y}) = \{ (\eta_1, \eta_2) : \eta_1 - f_i(\zeta^i) > M(\eta_1 - \zeta^i) \}$ be the cone with vertex $\hat{y}$ and opening $M$. Since $\Gamma_M(\hat{y}) \cap B(x_i, 4r) \subset D \cap B(x_i, 4r)$, we have

$$\delta(\xi) = |\xi - y| \geq \delta(\xi, \partial \Gamma_M(\hat{y})) = (M^2 + 1)^{-\frac{1}{2}}|\xi - f_i(\zeta^i)|.$$ 

Therefore

$$\int_{D \cap B(x_i, r)} \log \delta(\xi) |^r d\xi \leq \int_{D \cap B(x_i, r)} \left( \frac{1}{2} \log (M^2 + 1) - \log |\xi - f_i(\zeta^i)| \right)^r d\xi$$

$$\leq \int_{|\zeta^i| \leq 1} \left( \frac{1}{2} \log (M^2 + 1) - \log s_i \right)^r ds d\zeta^i < \infty.$$ 

Thus

$$\int_{D \cap \left( \bigcup_{i=1}^k B(x_i, r) \right)} |\log \delta(\xi)|^r d\xi < \infty.$$ 

Since $D$ is bounded and the distance between $D \setminus \bigcup_{i=1}^k B(x_i, r)$ and $\partial D$ is bounded away from zero by a positive constant, thus

$$\int_{D \setminus \bigcup_{i=1}^k B(x_i, r)} |\log \delta(\xi)|^r d\xi < \infty.$$ 

So $\log \delta \in L^p(D)$.

THEOREM 4.4. For any $\eta > 0$, we have

$$\int_D f^2 \log \frac{1}{1} d\xi \leq \eta \delta(f, f) + \beta(\eta) \| f \|_2^2,$$

with

$$\beta(\eta) = c_1 \eta^{-1/3} + c_2$$

for some positive constants $c_1$ and $c_2$. 
Proof. It is known that

$$\|u\|_{p_0} \leq c \sqrt{\delta(u, u)}, \quad \forall u \in \mathcal{F}$$

where $p_0$ is such that $1/p_0 = 1/2 - x/(2n)$ (see formula (1.5.20) of [18] or Theorem 1 on p. 119 of [28]). In particular, we have

$$\|u\|_{p_0} \leq c \sqrt{\delta(u, u)}, \quad \forall u \in \mathcal{F}. \quad (4.1)$$

It follows from Theorem 4.2 that for any $f \in \mathcal{F},$

$$\int_D f^2 \log \frac{1}{\phi_1} dx \leq c_1 \int_D f^2 \log \frac{1}{\delta} dx + c_2 \|f\|^2_2$$

Now applying the elementary inequality

$$ab \leq a^p + b^q, \quad a, b > 0, \quad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1$$

with $p = 4$ and $q = 4/3,$ we get (note that in the following constants $c_1, c_2$ may change their values from line to line)

$$\int_D f^2 \log \frac{1}{\phi_1} dx \leq \eta \int_D f^2 (\log \delta)^{1/3} dx + c_1 \eta^{-1/3} \|f\|^2_2 + c_2 \|f\|^2_2$$

where $p_0$ satisfies $1/p_0 = \frac{1}{2} - x/2n$ and $q_0$ is such that $2/p_0 + 1/q_0 = 1$. Therefore, using (4.1), we get

$$\int_D f^2 \log \frac{1}{\phi_1} dx \leq \eta \delta(f, f) + (\|\log \delta\|_{q_0}) \eta + c_1 \eta^{-1/3} + c_2 \|f\|^2_2.$$

By Lemma 4.3, $\|\log \delta\|_{q_0} < \infty$. So for $0 < \eta \leq 1,$ there exist positive constants $c_1$ and $c_2$ such that for $f \in \mathcal{F},$ we have

$$\int_D f^2 \log \frac{1}{\phi_1} dx \leq \eta \delta(f, f) + c_1 \|f\|^2_2.$$
where $\beta_1(\eta) = c_1 \eta^{-1/3} + c_2$. For $\eta > 1$, we have from the last inequality that for $f \in \mathcal{F}$,

$$
\int_D f^2 \log \frac{1}{\phi_1} \, dx \leq \mathcal{E}(f, f) + \beta_1(1) \| f \|_2^2 \\
\leq \eta \mathcal{E}(f, f) + \beta_1(1) \| f \|_2^2.
$$

Hence we can take $\beta(\eta) = c_1 \eta^{-1/3} + c_2$ such that for any $\eta > 0$,

$$
\int_D f^2 \log \frac{1}{\phi_1} \, dx \leq \eta \mathcal{E}(f, f) + \beta(\eta) \| f \|_2^2, \quad f \in \mathcal{F}.
$$

The proof is now complete. □

Combining Theorem 4.1 and Theorem 4.4 we get the following result.

**Theorem 4.5.** For any $\eta > 0$, and any $f \in \mathcal{F} \cap L^\infty(D, dx)$,

$$
\int_D f^2 \log \frac{1}{\phi_1} \, dx \leq \eta \mathcal{E}(f, f) + \beta_2(\eta) \| f \|_2^2, \quad f \in \mathcal{F},
$$

with

$$
\beta_2(\eta) = - \frac{n}{2a} \log \eta + c_1 \eta^{-1/3} + c_2
$$

for some positive constants $c_1$ and $c_2$.

**Theorem 4.6.** $L$ is intrinsically ultracontractive.

**Proof.** Let $m_0(dx) = \phi_1^2(x) \, dx$. Suppose that $(\mathcal{E}, \mathcal{F})$ is the Dirichlet form on $L^2(m_0) := L^2(D, dm_0)$ associated with the semigroup $\bar{P}_t$ whose integral kernel with respect to the measure $m_0$ is given by

$$
\frac{e^{-\lambda_1 t} P_t(x, y)}{\phi_1(x) \phi_1(y)}.
$$

Then

$$
\mathcal{F} = \{ h : h \phi_1 \in \mathcal{F} \}
$$

and

$$
\mathcal{E}(f, h) = \mathcal{E}(f \phi_1, h \phi_1) - \lambda_1 \int_D fh \, dm_0, \quad f, h \in \mathcal{F}.
$$
For any $h \in \mathcal{F} \cap L^\infty(D, dm_0)$, by putting $f = h\phi_1$ in (4.2) we get that for any $\eta > 0,$

$$
\int_D h^2 \log |h| \, dm_0 \leq \eta \tilde{E}(h, h) + (\beta_2(\eta) - \lambda_1) \| h \|_{L^2(m_0)}^2 + \| h \|_{L^2(m_0)}^2 \log \| h \|_{L^2(m_0)}.
$$

Therefore for $0 < \eta \leq 1,$

$$
\int_D h^2 \log |h| \, dm_0 \leq \eta \tilde{E}(h, h) + \beta_3(\eta) \| h \|_{L^2(m_0)}^2 + \| h \|_{L^2(m_0)}^2 \log \| h \|_{L^2(m_0)} \tag{4.3}
$$

with

$$
\beta_3(\eta) = -\frac{\eta}{2^x} \log \eta + c_1 \eta^{-1/3} + c_2 - \lambda_1.
$$

For $\eta > 1$, since $\tilde{E}$ is nonnegative and (4.3) holds for $\eta = 1$, we have for any $h \in \mathcal{F} \cap L^\infty(D, dm_0),$

$$
\int_D h^2 \log |h| \, dm_0 \leq \eta \tilde{E}(h, h) + \beta_3(1) \| h \|_{L^2(m_0)}^2 + \| h \|_{L^2(m_0)}^2 \log \| h \|_{L^2(m_0)}
$$

$$
\leq \eta \tilde{E}(h, h) + \beta_3(1) \| h \|_{L^2(m_0)}^2 + \| h \|_{L^2(m_0)}^2 \log \| h \|_{L^2(m_0)},
$$

(4.4)

Combining (4.3) and (4.4) we get that for any $\eta > 0$, and any $h \in \mathcal{F} \cap L^\infty(D, dm_0)$

$$
\int_D h^2 \log |h| \, dm_0 \leq \eta \tilde{E}(h, h) + A(\eta) \| h \|_{L^2(m_0)}^2 + \| h \|_{L^2(m_0)}^2 \log \| h \|_{L^2(m_0)},
$$

with

$$
A(\eta) = \begin{cases} 
\frac{\eta}{2^x} \log \eta + c_1 \eta^{-1/3} + c_2, & \text{if } \eta \leq 1, \\
 c_1 + c_2, & \text{if } \eta > 1.
\end{cases}
$$

for some positive constants $c_1$ and $c_2$. Thus by Corollary 2.2.8 of [14], $\tilde{P}_t$ is ultracontractive and therefore $L$ is intrinsically ultracontractive.
5. CONDITIONAL GAUGE THEOREM

In this section, we always assume that $D$ is a bounded $C^{1,1}$ domain and that $G_D(\cdot, \cdot)$ is the Green function of the symmetric $\alpha$-stable process $X^D$ on $D$. Recall that $L$ is the nonpositive definite infinitesimal generator of the killed $\alpha$-stable process on $D$. For $q \in K_{n, \alpha}$, let $u_q(t, x, y)$ be the kernel of the following Feynman–Kac semigroup

$$T_t f(x) = E^x \left[ e^t f(X_t) \mathbf{1}_{\{t < \tau_D\}} \right].$$

Note that the semigroup $T_t$ only depends on function $q$ through $q^1_D$ so we may assume that $q = 0$ off $D$.

For any $y \in D$, define the function

$$p^D_y(t, x, z) = \frac{G_D(x, y)}{G_D(z, y)} G_D(y, z), \quad t > 0, \quad x, z \in D.$$

Since $G_D(\cdot, y)$ is a strictly positive superharmonic function on $D$, $p^D_y$ is a transition density on $D \setminus \{y\}$. Thus it determines a Markov process on the state space $(D \setminus \{y\}) \cup \{\bar{y}\}$, where $\bar{y}$ is the cemetery point. The process is called $y$-conditioned symmetric $\alpha$-stable process in $D$ and its lifetime is defined to be $\zeta = \tau_{D \setminus \{y\}}$. The process remains at $\bar{y}$ in $[\zeta, \infty)$ on $[\zeta < \infty]$.

We continue to use $X_t$ to denote the generic random variable of the conditional process, but use $P^y$ and $E^y$ to denote its probability and expectation respectively.

**Theorem 5.1.** Suppose that $q \in K_{n, \alpha}$ is such that

$$\sup_{x, y \in D} \int_D \frac{G_D(x, z) |q(z)| G_D(z, y)}{G_D(x, y)} dz \leq \frac{1}{2}.$$

Then we have

$$e^{-1/2} G_D(x, y) \leq V_q(x, y) \leq 2 G_D(x, y), \quad (5.1)$$

where

$$V_q(x, y) = \int_0^\infty u_q(t, x, y) dt.$$

**Proof.** By Jensen's inequality and Khas'minskii's inequality, the conditional gauge function

$$F(x, y) = E^y \left[ e^\zeta \right], \quad x, y \in D,$$
must satisfy
\[ e^{-1/2} \leq \inf_{x, y \in D} F(x, y) \leq \sup_{x, y \in D} F(x, y) \leq 2. \tag{5.2} \]

Taking expectations on both sides of
\[ e_q(\tau_D \wedge t(y)) = 1 + \int_0^{\tau_D \wedge t(y)} q(X_s) \exp \left( \int_s^{\tau_D \wedge t(y)} q(X_t) \, dt \right) \, ds \]
we get that
\[ F(x, y) = 1 + E_x \left[ \int_0^{\tau_D \wedge t(y)} q(X_s) \exp \left( \int_s^{\tau_D \wedge t(y)} q(X_t) \, dt \right) \, ds \right] \]
\[ = 1 + E_x \left[ \int_0^{\tau_D \wedge t(y)} q(X_s) F(X_s, y) \, ds \right]. \]

The reason that we can interchange the order of integration in the last line is Theorem 1.3, since \( q \in K_{m,n} \) and \( F \) is bounded. Therefore
\[ F(x, y) = 1 + G_D(x, y)^{-1} \int_D G_D(x, w) q(w) F(w, y) G_D(w, y) \, dw \]
or equivalently
\[ G_D(x, y) F(x, y) = G_D(x, y) + \int_D G_D(x, w) q(w) F(w, y) G_D(w, y) \, dw. \]

Applying the operator \(-L\) in the \( x \) variable to both sides of the above equation, we get
\[ -(L + q(x))(G_D(x, y) F(x, y)) = \delta_{1, x}(x). \]

Therefore
\[ V_q(x, y) = F(x, y) G_D(x, y). \tag{5.3} \]

Thus the assertion of the theorem is true. \( \blacksquare \)
In what follows we always assume that the gauge function
\[ x \mapsto E^*[e^x \tau_D] \]
is finite at some \( x \in D \).

**Theorem 5.2.** \( L + q \) is intrinsically ultracontractive and there exists \( c > 0 \) such that
\[ V_q(x, y) \leq c G_D(x, y), \quad x, y \in D. \]

**Proof.** We prove this theorem in three steps. By Lemma 3.1 and Theorem 1.3, the function \( q \) can be decomposed as \( q = q_1 + q_2 \) with \( q_1 \) bounded and \( q_2 \in K_{n, \kappa} \) satisfying
\[ \sup_{x, y \in D} \int_D \frac{G_D(x, z) |q_D(z)| G_D(z, y)}{G_D(x, y)} dz \leq \frac{1}{2}. \]
Therefore by Theorem 5.1 we know that
\[ e^{-1/2} G_D(x, y) \leq V_q(x, y) \leq 2 G_D(x, y). \]

1. \( L + q_2 \) is intrinsically ultracontractive: Let \( \psi_1 \) be the ground state of \( L + q_2 \) and \( v_1 \) be the first eigenvalue of \( L + q_2 \). By (5.3) and Theorem 3.11 we know that \( v_1 < 0 \). Note that by Theorem 3.3, \( \psi_1 \in C_0(D) \). Moreover by Theorem 5.1 and Theorem 1.1,
\[ \psi_1(x) = -v_1^{-1} \int_0^\infty e^{t(L + q_1)} \psi_1(x) dt \geq c G_D \psi_1(x) \]
\[ \geq c \int_D \left( \frac{\delta(x)^{n-2} \delta(y)^{n/2}}{|x - y|^{n-\kappa}} \right) \psi_1(y) dy \geq c \delta(x)^{n/2}. \]
On the other hand, by Theorem 5.1, the boundedness of \( \psi_1 \) and Theorem 1.1 of [9]
\[ \psi_1(x) = -v_1^{-1} \int_0^\infty e^{t(L + q_1)} \psi_1(x) dt \leq c G_D \psi_1(x) \]
\[ \leq c \int_D \frac{\delta(x)^{n/2} \psi_1(y)}{|x - y|^{n - \kappa}} dy \leq c \delta(x)^{n/2}. \]
Thus there is a constant \( c > 1 \) such that
\[ c^{-1} \delta(x)^{n/2} \leq \psi_1(x) \leq c \delta(x)^{n/2}. \]
Let $\phi_1$ be the ground state of $L$. By Theorem 4.2, there is another constant $c > 1$ such that
\[ e^{-1} \delta(x)^{1/2} \leq \phi_1(x) \leq c \delta(x)^{1/2}. \] (5.5)
We know from Theorem 4.6 that $L$ is intrinsically ultracontractive. In fact we showed in Theorem 4.5 that for any $\eta > 0$ and $f \in \mathcal{F} \cap L^{\infty}(D, dx)$,
\[ \int_D f^2 \log \frac{|f|}{\phi_1} \, dx \leq \eta \varepsilon(f, f) + \beta(\eta) \|f\|_2^2 + \|f\|_2^2 \log \|f\|_2, \] (5.6)
with
\[ \beta(\eta) = -\frac{n}{\delta} \log \eta + c_1 \eta^{-1/3} + c_2 \] (5.7)
for some positive constants $c_1$ and $c_2$. Thus from (5.4), (5.5) and (5.6) we know that for any $\eta > 0$ and $f \in \mathcal{F} \cap L^{\infty}(D, dx)$,
\[ \int_D f^2 \log \frac{|f|}{\phi_1} \, dx \leq \eta \varepsilon(f, f) + \bar{\beta}(\eta) \|f\|_2^2 + \|f\|_2^2 \log \|f\|_2, \] (5.8)
where $\bar{\beta}(\eta)$ is the same as $\beta(\eta)$ except that the constant $c_2 > 0$ might have a different value.

Let $m_2(dx) = \psi_1^2(x) \, dx$. Suppose that $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is the Dirichlet form on $L^2(m_2)$ associated with the semigroup whose integral kernel with respect to the measure $m_2$ is given by
\[ e^{-\nu_1(t, x, y)} \psi_1(x) \psi_1(y). \]
Then
\[ \tilde{\mathcal{F}} = \{ h : h \psi_1 \in \mathcal{F} \} \]
and
\[ \tilde{\mathcal{E}}(f, h) = \varepsilon(f \psi_1, h \psi_1) - \int_D q_2 f \psi_1 h \psi_1 \, dx + v_1 \int_D fh \, dm_2. \]
Since $q_2 \in K_{n,s}$, by Theorem 3.2 of [27], there exists a constant $A > 0$ such that
\[ \int_D |q_2|^2 \, dx \leq \frac{1}{2} \varepsilon(u, u) + A \int_D u^2 \, dx, \quad u \in \mathcal{F}. \]
Thus
\[
\mathcal{E}(h, h) \geq \frac{1}{2} \mathcal{E}(h \psi_1, h \psi_1) - (A - \nu_1) \int_D h^2 \, dm_2.
\]

By putting \(f = h \psi_1\) in (5.8) we get that for any \(h \in \mathcal{F} \cap L^\infty(D, dm_2)\),
\[
\int_D h^2 \log |h| \, dm_2 \leq 2\eta \tilde{\mathcal{E}}(h, h) + (\tilde{B}(\eta) + 2(A - \nu_1)\eta) \int_D h^2 \, dm_2 + \|h\|_{L^2(m_2)}^2 \log \|h\|_{L^2(m_2)}.
\]

Therefore we have, for any \(0 < \eta \leq 1\) and \(h \in \mathcal{F} \cap L^\infty(D, dm_2)\),
\[
\int_D h^2 \log |h| \, dm_2 \leq \eta \tilde{\mathcal{E}}(h, h) + \beta_1(\eta) \int_D h^2 \, dm_2 + \|h\|_{L^2(m_2)}^2 \log \|h\|_{L^2(m_2)},
\]
(5.9)

where
\[
\beta_1(\eta) = -\frac{n}{2^2} \log \eta + c_1 \eta^{-1/3} + c_2
\]
for some constants \(c_1, c_2 > 0\). For \(\eta \geq 1\), since \(\tilde{\mathcal{E}}\) is nonnegative and (5.9) holds for \(\eta = 1\), we have for any \(h \in \mathcal{F} \cap L^\infty(D, dm_2)\),
\[
\int_D h^2 \log |h| \, dm_2 \leq \tilde{\mathcal{E}}(h, h) + \beta_1(1) \int_D h^2 \, dm_2 + \|h\|_{L^2(m_2)}^2 \log \|h\|_{L^2(m_2)}
\]
\[
\leq \eta \tilde{\mathcal{E}}(h, h) + \beta_1(1) \int_D h^2 \, dm_2 + \|h\|_{L^2(m_2)}^2 \log \|h\|_{L^2(m_2)},
\]
(5.10)

Combining (5.9) and (5.10) we get that for any \(\eta > 0\) and \(h \in \mathcal{F} \cap L^\infty(D, dm_2)\),
\[
\int_D h^2 \log |h| \, dm_2 \leq \eta \tilde{\mathcal{E}}(h, h) + A(\eta) \|h\|_{L^2(m_2)}^2 + \|h\|_{L^2(m_2)}^2 \log \|h\|_{L^2(m_2)},
\]
(5.11)

with
\[
A(\eta) = \begin{cases} 
-\frac{n}{2^2} \log \eta + c_1 \eta^{-1/3} + c_2, & \text{if } \eta \leq 1, \\
c_1 + c_2, & \text{if } \eta > 1,
\end{cases}
\]
for some positive constants $c_1$ and $c_2$. Therefore $L + q_2$ is intrinsically ultracontractive by Corollary 2.2.8 of [14].

2. $L + q$ is intrinsically ultracontractive: Since $L + q = (L + q_2) + q_1$, it follows from Theorem 3.4 of [15] that $L + q$ is intrinsic ultracontractive and the ground state $\varphi_1$ of $L + q$ is comparable to the ground state $\varphi_1'$ of $L + q_2$, i.e., there exists $c > 1$ such that

$$c^{-1}\varphi_1 \leq \varphi_1' \leq c\varphi_1. \quad (5.12)$$

But we need information more precise than the intrinsic ultracontractivity, see (5.17) below. From (5.8) and (5.12) we have that for any $\eta > 0$ and any $f \in \mathcal{F} \cap L^\infty(D, dx)$,

$$\int_D f^2 \log \frac{|f|}{\varphi_1} \, dx \leq \eta \varepsilon (f, f) + \beta_3(\eta) \| f \|_2^2 + \| f \|_2 \log \| f \|_2, \quad (5.13)$$

where $\beta_3(\eta)$ is the same $\overline{\beta}(\eta)$ except that the constant $c_2 > 0$ might have a different value.

Since the gauge function for $(D, q)$ is finite, the first eigenvalue $\mu_1$ of $L + q$ is negative. Suppose that $(\mathcal{F}, \mathcal{F})$ is the Dirichlet form on $L^2(m)$ with $m(dx) = \varphi_1^2 dx$ associated with the semigroup whose integral kernel with respect to the measure $m$ is given by

$$e^{-\mu_1 t}(x, y) \varphi_1(x) \varphi_1(y).$$

Then

$$\mathcal{F} = \{ f : f \varphi_1 \in \mathcal{F} \}$$

and

$$\mathcal{F}(f, h) = \varepsilon (f \varphi_1, h \varphi_1) - \int_D q f \varphi_1 h \varphi_1 \, dx + \mu_1 \int_D fh \, dm.$$ 

Since $q \in K_{a, s}$, by Theorem 3.2 of [27] there exists a constant $B > 0$ such that

$$\int_D |q| u^2 \, dx \leq \frac{1}{2} \varepsilon (u, u) + B \int_D u^2 \, dx, \quad u \in \mathcal{F}.$$ 

Thus

$$\mathcal{F}(h, h) \geq \frac{1}{2} \varepsilon (h \varphi_1, h \varphi_1) - (B - \mu_1) \int_D h^2 \, dm.$$
By putting \( f = h \rho_i \) in (5.13) we get that for \( h \in \mathcal{F} \cap L^\infty(D, dm) \),

\[
\int_D h^2 \log |h| \ dm \leq 2\eta \mathcal{F}(h, h) + (\beta(\eta) + 2(B - \mu_i) \eta) 
\times \int_D h^2 \ dm + \|h\|_L^2 \log \|h\|_L^2.
\]

(5.14)

Therefore we have, for any \( 0 < \eta \leq 1 \) and any \( h \in \mathcal{F} \cap L^\infty(D, dm) \),

\[
\int_D h^2 \log |h| \ dm \leq \eta \mathcal{F}(h, h) + \beta(\eta) \int_D h^2 \ dm + \|h\|_L^2 \log \|h\|_L^2,
\]

(5.15)

where

\[
\beta(\eta) = -\frac{n}{2\alpha} \log \eta + c_1 \eta^{-1/3} + c_2,
\]

for some constants \( c_1, c_2 > 0 \). For \( \eta > 1 \), since \( \mathcal{F} \) is nonnegative and (5.15) holds for \( \eta = 1 \), we have for any \( h \in \mathcal{F} \cap L^\infty(D, dm) \),

\[
\int_D h^2 \log |h| \ dm \leq \eta \mathcal{F}(h, h) + \beta(1) \int_D h^2 \ dm + \|h\|_L^2 \log \|h\|_L^2.
\]

(5.16)

Combining (5.15) and (5.16) we get that for any \( \eta > 0 \) and any \( h \in \mathcal{F} \cap L^\infty(D, dm) \),

\[
\int_D h^3 \log |h| \ dm \leq \eta \mathcal{F}(h, h) + B(\eta) \|h\|_L^2 + \|h\|_L^2 \log \|h\|_L^2,
\]

(5.17)

with

\[
B(\eta) = \begin{cases} 
-\frac{n}{2\alpha} \log \eta + c_1 \eta^{-1/3} + c_2, & \text{if } \eta \leq 1, \\
(c_1 + c_2), & \text{if } \eta > 1,
\end{cases}
\]

for some positive constants \( c_1 \) and \( c_2 \).
3. **There is a constant** \( c > 0 \) **such that** \( V_q \leq cG_D \): **By Corollary 2.2.8** of [14], we have

\[
e^{-\rho(t)} \frac{u(t, x, y)}{\varphi_1(x) \varphi_1(y)} \leq e^{2N(t/2)},
\]

where

\[
N(t) = \frac{1}{t} \int_0^t \beta(\eta) \, d\eta.
\]

Thus there is a constant \( c > 0 \) such that for \( t \geq 1 \) we have

\[
u(t, x, y) \leq ce^{n/q}(x) \varphi_1(y).
\]

Therefore by Theorem 1.1

\[
\int_{-1}^\infty u(t, x, y) \, dt \leq \epsilon \varphi_1(x) \varphi_1(y) \leq c\delta(x)^{n/2} \delta(y)^{n/2} \leq cG_D(x, y).
\]

Since \( q_1 \) is bounded, it follows that

\[
u(t, x, y) \leq e^{\|q_1\|_\infty} u_{q_1}(t, x, y), \quad \forall t > 0, x, y \in D.
\]

Thus by Theorem 5.1,

\[
\int_0^2 u(t, x, y) \, dt \leq e^{2\|q_1\|_\infty} u_{q_1}(t, x, y) \, dt \leq e^{2\|q_1\|_\infty} V_{q_1}(x, y) \leq cG_D(x, y).
\]

Hence there is a constant \( c \geq 0 \) such that

\[
V_q(x, y) \leq cG_D(x, y) \quad \forall x, y \in D.
\]

**Theorem 5.3.** For all \( x, y \in D \) with \( x \neq y \),

\[
V_q(x, y) = G_D(x, y) + \int_D V_q(x, u) \, q(u) \, G_D(u, y) \, du \quad (5.18)
\]

\[
V_q(x, y) = G_D(x, y) + \int_D G_D(x, u) \, q(u) \, V_q(u, y) \, du. \quad (5.19)
\]
Proof. It follows from Theorem 1.3, Theorem 5.2 and the assumption \( q \in K_{n,s} \) that the family of functions
\[
\left\{ \frac{V_q(x, \cdot) |q(\cdot)|}{G_D(\cdot, y)} : x, y \in D \right\}
\]
is uniformly integrable \( D \). On the other hand, for each \( u \in D \), the function
\[
(x, y) \mapsto \frac{V_q(x, u) q(u) G_D(u, y)}{G_D(x, y)}
\]
is continuous except possibly at \( x = u \) or \( y = u \). Therefore, the integral on the right side of (5.18) is continuous in \( (x, y) \in D \times D \) with \( x \neq y \), consequently, both side of (5.18) are continuous in \( (x, y) \in D \times D \) with \( x \neq y \).

For any \( f \in C_0(D) \), we have \( G_D f \in C_0(D) \). Therefore by the gauge theorem we have \( V_q(qG_D f) \in C_0(D) \). Hence by the Markov property and Fubini’s theorem, we have
\[
V_q(qG_D f)(x) = E^x E[Y_0 \int_0^t e_q(t) q(X_s) E_{X(t)} \int_0^{\tau_0} f(X_s) ds \ dt]
= E^x E[Y_0 \int_0^t e_q(t) q(X_s) \left( \int_0^{\tau_0} f(X_s) ds \ dt \right) ds]
= E^x \int_0^t f(X_s) (e_q(s) - 1) ds = V_q f(x) - G_D f(x).
\]
Therefore (5.18) holds for almost all \( (x, y) \in D \times D \) and consequently, by the continuity, (5.18) holds for all \( (x, y) \in D \times D \) with \( x \neq y \). (5.19) follows from (5.18) by the symmetry of \( G_D(\cdot, \cdot) \) and \( V_q(\cdot, \cdot) \).

**Theorem 5.4.** For all \( (x, y) \in D \times D \) with \( x \neq y \),
\[
E^x_+ e_q(\zeta) = 1 + G_D(x, y)^{-1} \int_D V_q(x, w) q(w) G_D(w, y) dw.
\]

Proof. It follows from Theorem 1.3 and Theorem 5.2 that for any \( (x, y) \in D \times D \), \( x \neq y \),
\[
G_D(x, y)^{-1} \int_D V_q(x, w) |q(w)| G_D(w, y) dw < \infty.
\]
By Fubini's theorem we have

\[
\begin{aligned}
E^*_x \int_0^\zeta e_q(t) |q(X_t)| \, dt &= \int_0^\zeta E^*_y [e_q(t) |q(X_t)|; t < \zeta] \, dt \\
&= G_D(x, y)^{-1} \int_0^\infty E^*_x [e_q(t) |q(X_t)|; t < \tau_D] \, dt \\
&= G_D(x, y)^{-1} \int_0^\infty \int_D u(t, x, y) |q(w)| G_D(w, y) \, dw \, dt \\
&= G_D(x, y)^{-1} \int_D V_q(x, w) |q(w)| G_D(w, y) \, dw < \infty.
\end{aligned}
\]

Therefore

\[
E^*_x [e_q(\zeta)] = 1 + E^*_y \left[ \int_0^\zeta e_q(t) q(X_t) \, dt \right] = 1 + G_D(x, y)^{-1} \int_D V_q(x, w) q(w) G_D(w, y) \, dw.
\]

**Theorem 5.5.** For all \((x, y) \in D \times D\) with \(x \neq y\),

\[
E^*_x [e_q(\zeta)] = \frac{V_q(x, y)}{G_D(x, y)}.
\]

**Proof.** This follows immediately from Theorem 5.3 and Theorem 5.4.

The following two results are the conditional gauge theorems.

**Theorem 5.6.** There exist \(c > 1\) such that

\[
c^{-1} \leq \inf_{x, y \in D} E^*_x [e_q(\zeta)] \leq \sup_{x, y \in D} E^*_x [e_q(\zeta)] \leq c.
\]

**Proof.** The inequality on the right follows from Theorem 5.2 and Theorem 5.5. For \(x, y \in D\),

\[
E^*_x [e_q(\zeta)] = \frac{V_q(x, y)}{G_D(x, y)}.
\]
It follows from Theorem 1.3 that the family of functions
\[ \{ G_D(x, \cdot) \mid q(\cdot) \mid G_D(\cdot, y) \mid x, y \in D \} \]
is uniformly integrable and therefore
\[ M = \sup_{x, y \in D} E_x^y \left[ \int_0^\zeta |q(X_s)| \, ds \right] < \infty. \]

Now by Jensen's inequality, for all \( x, y \in D \),
\[ E_x^y [ e_\zeta ] \geq \exp \left( E_x^y \left[ \int_0^\zeta q(X_s) \, ds \right] \right) \geq e^{-M}. \]
The proof is now complete.

Combining Theorem 5.5 and Theorem 5.6 we immediately get the following result:

**Theorem 5.7.** There exists a constant \( c > 1 \) such that
\[ c^{-1} G_D(x, y) \leq V_q(x, y) \leq c G_D(x, y). \]
That is, the Green function of \( D \) with respect to \( L + q \) is comparable to the Green function of \( D \) with respect to \( L \).

For any \( z \in \partial D \), we define the \( z \)-conditioned symmetric \( \alpha \)-stable process on the state space \( D \cup \{ z \} \) by the transition density
\[ p_D^z(t, x, y) = \frac{1}{K_D(x, z)} p_D^0(t, x, y) K_D(y, z), \quad t > 0, \quad x, y \in D, \]
where \( K_D(x, z) \) is the Poisson kernel of the \( \alpha \)-stable process for \( D \). The lifetime of the conditional process is \( \zeta = \tau_D \). We are going to use \( P_D^z \) and \( E_D^z \) to denote the probability and expectation, respectively, for the \( z \)-conditioned symmetric \( \alpha \)-stable process in \( D \).
Theorem 5.8. There exists a constant \( c > 1 \) such that

\[
\inf_{(x, z) \in D \times D^c} E^x[e_q(\tau_D)] \leq \sup_{(x, z) \in D \times D^c} E^x[e_q(\tau_D)] \leq c.
\]

Proof. By using the same arguments as in the proof of Theorem 5.4, we can get

\[
E^x[e_q(\tau_D)] = 1 + \frac{1}{K_D(x, z)} \int_D V_q(x, w) q(w) K_D(w, z) \, dw.
\]

Since by Theorem 1.4 of [9]

\[
K_D(x, z) = \int_D \frac{G_D(x, u)}{|u-z|^{n+\alpha}} \, du,
\]

we get

\[
E^x[e_q(\tau_D)] = 1 + \frac{1}{K_D(x, z)} \int_D V_q(x, w) q(w) \frac{1}{|v-z|^{n+\alpha}} \int_D G_D(x, v) \cdot \int_D V_q(x, w) q(w) \, dv \, dw.
\]

Therefore the assertion of this theorem follows from Theorem 5.6. \( \square \)

Suppose that \( K_q(x, z), x \in D, z \in D^c \), is the Poisson kernel of \( D \) with respect to \( L + q \). That is, for any \( \phi \geq 0 \) on \( D^c \), any \( x \in D \) and \( z \in D^c \), \( K_q(x, z) \) satisfies the following relation

\[
E^x[e_q(\tau_D) \phi(X_{\tau_D})] = \int_{D^c} K_q(x, z) \phi(z) \, dz.
\]

The next theorem shows that the z-conditioned symmetric \( \alpha \)-stable process is the symmetric \( \alpha \)-stable process conditioned to exit \( D \) at \( z \). For \( t > 0 \), let \( F_t = \sigma(X_s, 0 \leq s \leq t) \) and \( F_{0+} = \bigcap_{t > 0} F_t \). Recall that \( \mathcal{F}_{0+} \) is the \( \sigma \)-field generated by \( \mathcal{F}_{0+} \) and the class of sets \( \{ A_t \cap [ t < \tau_D ] : t > 0, A_t \in \mathcal{F} \} \).
Theorem 5.9. For any $\mathcal{F}_t$-measurable random variable $\xi \geq 0$ and $x \in D$, we have

$$E^x(\xi|X_t) = E^x_{X_t}[\xi].$$

Proof. Same as that of Proposition 5.11 of [12].

Theorem 5.10. There exists a constant $c > 1$ such that

$$c^{-1}K_p(x, z) \leq K_q(x, z) \leq cK_p(x, z), \quad \forall x \in D, \quad z \in D^c.$$

That is, the Poisson kernel of $D$ with respect to $L+q$ is comparable to the Poisson kernel of $D$ with respect to $L$.

Proof. It is easy to see that $e_t(\tau_D)$ is $\mathcal{F}_t$-measurable. So by Theorem 5.9 for any Borel measurable function $\varphi \geq 0$ on $D^c$,

$$E^x[e_t(\tau_D) \varphi(X_t)] = \int_{D^c} E^x_z[e_t(\tau_D)K_p(x, z)] \varphi(z) dz,$$

and thus

$$K_p(x, z) = E^x_z[e_t(\tau_D)]K_p(x, z).$$

Now the assertion follows from the Theorem 5.8.

REFERENCES


