Outline

1. General Info
2. 9.4 The Local Time of Linear Brownian Motion
HW7 is due Friday, 12/11, at noon. Please submit your homework via the course Moodle page. Make sure that your HW is uploaded successfully.

HW6 has been graded now.
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1. General Info

2. 9.4 The Local Time of Linear Brownian Motion
Throughout this section, $B = (B_t)_{t \geq 0}$ is a 1-dim Brownian motion started from 0 and $(\mathcal{F}_t)$ is the (completed) canonical filtration of $B$.

The following theorem, which is known as Trotter’s theorem, is essentially a restatement of the results of the previous sections in the special case of 1-dim Brownian motion. Still the importance of the result justifies this repetition. We write $\text{supp}(\mu)$ for the topological support of a finite measure $\mu$ on $\mathbb{R}_+$. 
Throughout this section, \( B = (B_t)_{t \geq 0} \) is a 1-dim Brownian motion started from 0 and \( (\mathcal{F}_t) \) is the (completed) canonical filtration of \( B \).

The following theorem, which is known as Trotter’s theorem, is essentially a restatement of the results of the previous sections in the special case of 1-dim Brownian motion. Still the importance of the result justifies this repetition. We write \( \text{supp}(\mu) \) for the topological support of a finite measure \( \mu \) on \( \mathbb{R}_+ \).
**Theorem 9.12 (Trotter)**

There exists a (unique) process $(L^a_t(B))_{a \in \mathbb{R}, t \geq 0}$, whose sample paths are continuous functions of the pair $(a, t)$, such that, for every fixed $a \in \mathbb{R}$, $(L^a_t(B))_{t \geq 0}$ is an increasing process, and, a.s. for every $t \geq 0$, for every non-negative measurable function $\phi$ on $\mathbb{R}$,

$$\int_0^t \phi(B_s)ds = \int_{\mathbb{R}} \phi(a)L^a_t(B)da.$$

Furthermore, a.s. for every $a \in \mathbb{R}$,

$$\text{supp}(d_sL^a_s(B)) \subset \{s \geq 0 : B_s = a\}. \quad (1)$$

and this inclusion is an equality with probability one if $a$ is fixed.
Proof of Theorem 9.12

The first assertion follows by applying Theorem 9.4 and Corollary 9.7 to \( X = B \), noting that \( \langle B, B \rangle_t = t \). We have already seen that the inclusion (1) holds with probability one if \( a \) is fixed, hence simultaneously for all rationals, a.s. A continuity argument allows us to get that (1) holds simultaneously for all \( a \in \mathbb{R} \) outside a single set of probability zero. Indeed, suppose that for some \( a \in \mathbb{R} \) and \( 0 \leq s < t \), we have \( L^a_t(B) > L^a_s(B) \) and \( B_r \neq a \) for every \( r \in [s, t] \).

Then we can find a rational \( b \in \mathbb{R} \) sufficiently close to \( a \) such that the same properties hold when \( a \) is replaced by \( b \), giving a contradiction.

Now let’s verify that (1) is an a.s. equality if \( a \in \mathbb{R} \) is fixed. So let us fix \( a \in \mathbb{R} \) and for every rational \( q \geq 0 \), define

\[ H_q = \inf \{ t \geq q : B_t = a \}. \]

Our claim will follow if we can verify that a.s. for every \( \epsilon > 0 \),

\[ L^a_{H_q+\epsilon} > L^a_{H_q}. \]

Using the strong Markov property at time \( H_q \), it suffices
to prove that, if $B'$ is a 1-dim Brownian motion started from $a$, we have $L^a_\epsilon(B') > 0$, for every $\epsilon > 0$, a.s. Clearly we can take $a = 0$. We then observe that we have

$$L^0_\epsilon(B) = \sqrt{\epsilon} L^0_1(B), \quad \text{in distribution},$$

by an easy scaling argument (use for instance the approximations of the previous section). Also $\mathbb{P}(L^0_1(B) > 0) > 0$ since $\mathbb{E}[L^0_1(B)] = \mathbb{E}[|B_1|]$ by Tanaka’s formula. An application of Blumenthal’s zero-one law to the event

$$A := \bigcap_{n=1}^\infty \{L^{0}_{2^{-n}}(B) > 0\} = \lim_{n \to \infty} \uparrow \{L^{0}_{2^{-n}}(B) > 0\}$$

completes the proof.

Theorem 9.12 remains true with a similar proof for an arbitrary (possibly random) initial value $B_0$. 
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completes the proof.

Theorem 9.12 remains true with a similar proof for an arbitrary (possibly random) initial value $B_0$. 
We now turn to distributional properties of local times of Brownian motion.

**Proposition 9.13**

(i) Let $a \in \mathbb{R} \setminus \{0\}$ and $T_a = \inf\{t \geq 0 : B_t = a\}$. Then $L_{T_a}^0(B)$ has an exponential distribution with mean $2|a|$.

(ii) Let $a > 0$ and $U_a = \inf\{|B_t| = a\}$. Then $L_{U_a}^0(B)$ has an exponential distribution with mean $a$. 
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Proof of Proposition 9.13

(i) By simple scaling and symmetry arguments, it is enough to consider the case $a = 1$. We then observe that $L^0_\infty(B) = \infty$ a.s. Indeed, the scaling argument of the preceding proof shows that $L^0_\infty(B)$ has the same distribution as $\lambda L^0_\infty(B)$, for any $\lambda > 0$, and we have also seen that $L^0_\infty(B) > 0$ a.s. Fix $s > 0$ and define

$$\tau = \inf\{t \geq 0 : L^0_t(B) \geq s\},$$

so that $\tau$ is a stopping time of the filtration $(\mathcal{F}_t)$. Furthermore, $B_\tau = 0$ by the support property of local time. By the strong Markov property,

$$B'_t := B_{\tau+t}$$

is a Brownian motion started from 0, which is also independent of $\mathcal{F}_\tau$. Proposition 9.9 gives, for every $t \geq 0$,

$$L^0_t(B') = L^0_{\tau+t}(B) - s.$$
Proof of Proposition 9.13 (cont)

On the event \( \{L^0_{T_1}(B) \geq s\} = \{\tau \leq T_1\} \), we thus have

\[
L^0_{T_1}(B) - s = L^0_{T_1 - \tau}(B') = L^0_{T'_1}(B'),
\]

where \( T'_1 = \inf\{t \geq 0 : B'_t = 1\} \). Since the event \( \{\tau \leq T_1\} \) is \( \mathcal{F}_\tau \)-measurable and \( B' \) is independent of \( \mathcal{F}_\tau \), we get that the conditional distribution of \( L^0_{T_1}(B) - s \) knowing that \( L^0_{T_1}(B) \geq s \) is the same as the unconditional distribution of \( L^0_{T_1}(B) \). This implies that the distribution of \( L^0_{T_1}(B) \) is exponential.

Finally, Tanaka’s formula shows that \( \frac{1}{2} \mathbb{E}[L^0_{t \wedge T_1}(B)] = \mathbb{E}[(B_{t \wedge T_1})^+] \). As \( t \to \infty \), \( \mathbb{E}[L^0_{t \wedge T_1}(B)] \) converges to \( \mathbb{E}[L^0_{T_1}(B)] \) by monotone convergence and \( \mathbb{E}[(B_{t \wedge T_1})^+] \) converges to \( \mathbb{E}[(B_{T_1})^+] \) by dominated convergence, since \( 0 \leq (B_{t \wedge T_1})^+ \leq 1 \). This shows that \( \mathbb{E}[L^0_{T_1}(B)] = 2 \), as desired.
Proof of Proposition 9.13 (cont)

(ii) The argument is exactly similar. We now use Tanaka’s formula for absolute value to verify that $\mathbb{E}[L^0_{U_a}(B)] = a$.

Remark

One can give an alternative proof of the proposition using stochastic calculus. To get (ii), for instance, use Ito’s formula to verify that, for every $\lambda > 0$,

$$(1 + \lambda|B_t|) \exp(-\lambda L^0_t(B))$$

is a continuous local martingale, which is bounded on $[0, U_a]$. An application of the optional stopping theorem then shows that $\mathbb{E}[\exp(-\lambda L^0_t(B))] = (1 + \lambda a)^{-1}$.

For every $t \geq 0$, we define

$$S_t := \sup_{s \leq t} B_s, \quad l_t := \inf_{s \leq t} B_s.$$
Proof of Proposition 9.13 (cont)

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For every $t \geq 0$, we define

$$S_t := \sup_{s \leq t} B_s, \quad I_t := \inf_{s \leq t} B_s.$$
Theorem 9.14 (Lévy)

The two processes \((S_t, S_t - B_t)_{t \geq 0}\) and \((L^0_t(B), |B_t|)_{t \geq 0}\) have the same distribution.

Remark

By an obvious symmetry argument, the pair \((-I_t, B_t - I_t)_{t \geq 0}\) also has the same distribution as \((S_t, S_t - B_t)_{t \geq 0}\).

Proof of Theorem 9.14

By Tanaka’s formula, for every \(t \geq 0\),

\[
|B_t| = -\beta_t + L^0_t(B),
\]

where

\[
\beta_t = -\int_0^t \text{sgn}(B_s) dB_s.
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By an obvious symmetry argument, the pair \((-l_t, B_t - l_t)_{t \geq 0}\) also has the same distribution as \((S_t, S_t - B_t)_{t \geq 0}\).

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Proof of Theorem 9.14 (cont)

Since $\langle \beta, \beta \rangle_t = t$, Theorem 5.12 ensures that $\beta$ is a 1-dim Brownian motion started from 0. We then claim that, for every $t \geq 0$,

$$L^0_t(B) = \sup\{\beta_s : s \leq t\}.$$  

The fact that $L^0_t(B) = \sup\{\beta_s : s \leq t\}$ is immediate since (2) shows that $L^0_t(B) \geq L^0_s(B) \geq \beta_s$, for every $s \in [0, t]$. To get the reverse inequality, write $\gamma_t$ for the last zero of $B$ before time $t$. By the support property of local time, $L^0_t(B) = L^0_{\gamma_t}(B)$ and using (2),

$$L^0_{\gamma_t}(B) = \beta_{\gamma_t} \leq \sup\{\beta_s : s \leq t\}.$$  

We have thus proved a.s.

$$(L^0_t(B), |B_t|)_{t \geq 0} = (\sup\{\beta_s : s \leq t\}, \sup\{\beta_s : s \leq t\} - \beta_t)_{t \geq 0},$$

and since $(\beta_s)_{s \geq 0}$ and $(B_s)_{s \geq 0}$ have the same distribution, the pair in the right-hand side has the same distribution as $(S_t, S_t - B_t)_{t \geq 0}$. 
Theorem 9.14 has several interesting consequences. For every $t \geq 0$, $S_t$ has the same law as $|B_t|$ (Theorem 2.21), and thus the same holds for $L_0^t(B)$. From the explicit formula (2.2) for the density of $(S_t, B_t)$, we also get the density of the pair $(L_0^t(B), B_t)$.

For every $s \geq 0$, define

$$\tau_s := \inf\{ t \geq 0 : L_0^t(B) > s \}.$$ 

The process $(\tau_s)_{s \geq 0}$ is called the inverse local time (at 0) of the Brownian motion $B$. By construction, $(\tau_s)_{s \geq 0}$ has cadlag increasing sample paths. From L’evy Theorem 9.14, one gets that

$$(\tau_s)_{s \geq 0} = (\tilde{T}_s)_{s \geq 0}, \quad \text{in distribution},$$

where, for every $s \geq 0$, $\tilde{T}_s = \inf\{ t \geq 0 : B_t > s \}$. 
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where, for every $s \geq 0$, $\tilde{T}_s = \inf\{t \geq 0 : B_t > s\}$. 
The same application of the strong Markov property as in the proof of Proposition 9.13 shows that \((\tau_s)_{s \geq 0}\) has stationary independent increments. Furthermore, using the invariance of Brownian motion under scaling, we have for every \(\lambda > 0\),

\[
(\tau_{\lambda s})_{s \geq 0} = (\lambda^2 \tau_s)_{s \geq 0}, \quad \text{in distribution.}
\]

The preceding properties can be summarized by saying that \((\tau_s)_{s \geq 0}\) is a stable subordinator with index 1/2 (a subordinator is a Lévy process with non-decreasing sample paths).

The interest of considering the process \((\tau_s)_{s \geq 0}\) comes in part from the following proposition.
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Proposition 9.15

We have a.s.

\[ \{ t \geq 0 : B_t = 0 \} = \{ \tau_s : s \geq 0 \} \cup \{ \tau_{s^-} : s \in D \}, \]

where \( D \) is the countable set of jump times of \((\tau_s)_{s \geq 0}\).

Proof

We know from (1) that a.s.

\[ \text{supp}(d_t L_0^0(B)) \subset \{ t \geq 0 : B_t = 0 \}. \]

It follows that any time \( t \) of the form \( t = \tau_s \) or \( t = \tau_{s^-} \) must belong to the zero set of \( B \). Conversely, recalling that (1) is an a.s. equality for \( a = 0 \), we also get that, a.s. for every \( t \) such that \( B_t = 0 \), we have either \( L_{t+\epsilon}^0 > L_t^0(B) \) for every \( \epsilon > 0 \), or, if \( t > 0 \), \( L_{t-\epsilon}^0 < L_t^0(B) \) for every \( \epsilon > 0 \) with \( \epsilon < t \) (or both simultaneously), which implies that we have \( t = \tau_{L_t^0(B)} \) or \( t = \tau_{L_t^0(B)-} \).
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It follows that any time $t$ of the form $t = \tau_s$ or $t = \tau_{s-}$ must belong to the zero set of $B$. Conversely, recalling that (1) is an a.s. equality for $a = 0$, we also get that, a.s. for every $t$ such that $B_t = 0$, we have either $L_{t+\epsilon}^0 > L_t^0(B)$ for every $\epsilon > 0$, or, if $t > 0$, $L_{t-\epsilon}^0 < L_t^0(B)$ for every $\epsilon > 0$ with $\epsilon < t$ (or both simultaneously), which implies that we have $t = \tau_{L_t^0(B)}$ or $t = \tau_{L_t^0(B)-}$. 
As a consequence of Proposition 9.15, the connected components of the complement of the zero set \( \{ t \geq 0 : B_t = 0 \} \) are exactly the intervals \( (\tau_{s-}, \tau_s) \) for \( s \in D \). These connected components are called the excursion intervals (away from 0). For every \( s \in D \), the associated excursion is defined by

\[
e_s(t) := B_{(\tau_{s-} + t) \wedge \tau_s}, \quad t \geq 0.
\]

The goal of excursion theory is to describe the distribution of the excursion process, that is, of the collection \( (e_s)_{s \in D} \). This theory is very useful.