Math 562 Fall 2020

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Outline

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HW7 is due Friday, 12/11, at noon. Please submit your homework via the course Moodle page. Make sure that your HW is uploaded successfully.

HW6 has been graded now.

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Throughout this section, $B = (B_t)_{t \geq 0}$ is a 1-dim Brownian motion started from 0 and $(\mathcal{F}_t)$ is the (completed) canonical filtration of B.

The following theorem, which is known as Trotter's theorem, is essentially a restatement of the results of the previous sections in the special case of 1-dim Brownian motion. Still the importance of the result justifies this repetition. We write $\mathrm{supp}(\mu)$ for the topological support of a finite measure μ on $\mathbb{R}_+$.

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Theorem 9.12 (Trotter)

There exists a (unique) process $(L_t^a(B))_{a\in\mathbb{R},t\geq 0}$, whose sample paths are continuous functions of the pair (a,t), such that, for every fixed $a\in\mathbb{R}$, $(L_t^a(B))_{t\geq 0}$ is an increasing process, and, a.s. for every $t\geq 0$, for every non-negative measurable function ϕ on $\mathbb{R}$,

$$\int_0^t \phi(B_s)ds = \int_{\mathbb{R}} \phi(a)L_t^a(B)da.$$

Furthermore, a.s. for every $a\in\mathbb{R}$,

$$\mathrm{supp}(d_s L_s^a(B)) \subset \{s\geq 0 : B_s = a\}. \tag{1}$$

and this inclusion is an equality with probability one if a is fixed.

Proof of Theorem 9.12

The first assertion follows by applying Theorem 9.4 and Corollary 9.7 to $X = B$, noting that $\langle B, B \rangle_t = t$. We have already seen that the inclusion (1) holds with probability one if a is fixed, hence simultaneously for all rationals, a.s. A continuity argument allows us to get that (1) holds simultaneously for all $a \in \mathbb{R}$ outside a single set of probability zero. Indeed, suppose that for some $a \in \mathbb{R}$ and $0 \leq s < t$, we have $L_t^a(B) > L_s^a(B)$ and $B_r \neq a$ for every $r \in [s, t]$. Then we can find a rational $b \in \mathbb{R}$ sufficiently close to a such that the same properties hold when a is replaced by b, giving a contradiction.

Now let's verify that (1) is an a.s. equality if $a \in \mathbb{R}$ is fixed. So let us fix $a \in \mathbb{R}$ and for every rational $q \geq 0$, define

$$H_q = \inf\{t \geq q : B_t = a\}.$$

Our claim will follow if we can verify that a.s. for every $\epsilon > 0$, $L^a_{H_q+\epsilon} > L^a_{H_q}$. Using the strong Markov property at time H_q, it suffices

Proof of Corollary Theorem 9.12 (cont)

to prove that, if B' is a 1-dim Brownian motion started from a, we have $L^a_\epsilon(B') > 0$, for every $\epsilon > 0$, a.s. Clearly we can take $a = 0$. We then observe that we have

$$L^0_\epsilon(B) = \sqrt{\epsilon} L^0_1(B), \qquad \text{in distribution,}$$

by an easy scaling argument (use for instance the approximations of the previous section). Also $\mathbb{P}(L^0_1(B) > 0) > 0$ since $\mathbb{E}[L^0_1(B)] = \mathbb{E}[|B_1|]$ by Tanaka's formula. An application of Blumenthal's zero-one law to the event

$$A := \cap_{n=1}^{\infty} \{L^0_{2^{-n}}(B) > 0\} = \lim_{n\to\infty} \uparrow \{L^0_{2^{-n}}(B) > 0\}$$

completes the proof.

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We now turn to distributional properties of local times of Brownian motion.

Proposition 9.13

(i) Let $a \in \mathbb{R} \setminus \{0\}$ and $T_a = \inf\{t \geq 0 : B_t = a\}$. Then $L^0_{T_a}(B)$ has an exponential distribution with mean $2|a|$.

(ii) Let $a > 0$ and $U_a = \inf\{|B_t| = a\}$. Then $L^0_{U_a}(B)$ has an exponential distribution with mean a.

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(ii) Let $a > 0$ and $U_a = \inf\{|B_t| = a\}$. Then $L^0_{U_a}(B)$ has an exponential distribution with mean a.

Proof of Proposition 9.13

(i) By simple scaling and symmetry arguments, it is enough to consider the case $a = 1$. We then observe that $L^0_\infty(B) = \infty$ a.s. Indeed, the scaling argument of the preceding proof shows that $L^0_\infty(B)$ has the same distribution as $\lambda L^0_\infty(B)$, for any $\lambda > 0$, and we have also seen that $L^0_\infty(B) > 0$ a.s. Fix $s > 0$ and define

$$\tau = \inf\{t \geq 0 : L^0_t(B) \geq s\},$$

so that τ is a stopping time of the filtration $(\mathcal{F}_t)$. Furthermore, $B_\tau = 0$ by the support property of local time. By the strong Markov property,

$$B'_t := B_{\tau+t}$$

is a Brownian motion started from 0, which is also independent of $\mathcal{F}_\tau$. Proposition 9.9 gives, for every $t \geq 0$,

$$L^0_t(B') = L^0_{\tau+t}(B) - s.$$

Proof of Proposition 9.13 (cont)

On the event $\{L^0_{T_1}(B) \geq s\} = \{\tau \leq T_1\}$, we thus have

$$L^0_{T_1}(B) - s = L^0_{T_1-\tau}(B') = L^0_{T_1'}(B'),$$

where $T_1' = \inf\{t \geq 0 : B_t' = 1\}$. Since the event $\{\tau \leq T_1\}$ is $\mathcal{F}_\tau$-measurable and B' is independent of $\mathcal{F}_\tau$, we get that the conditional distribution of $L^0_{T_1}(B) - s$ knowing that $L^0_{T_1}(B) \geq s$ is the same as the unconditional distribution of $L^0_{T_1}(B)$. This implies that the distribution of $L^0_{T_1}(B)$ is exponential.

Finally, Tanaka's formula shows that $\frac{1}{2}\mathbb{E}[L^0_{t\wedge T_1}(B)] = \mathbb{E}[(B_{t\wedge T_1})^+]$. As $t \to \infty$, $\mathbb{E}[L^0_{t\wedge T_1}(B)]$ converges to $\mathbb{E}[L^0_{T_1}(B)]$ by monotone convergence and $\mathbb{E}[(B_{t\wedge T_1})^+]$ converges to $\mathbb{E}[(B_{T_1})^+]$ by dominated convergence, since $0 \leq (B_{t\wedge T_1})^+ \leq 1$. This shows that $\mathbb{E}[L^0_{T_1}(B)] = 2$, as desired.

Proof of Proposition 9.13 (cont)

(ii) The argument is exactly similar. We now use Tanaka's formula for absolute value to verify that $\mathbb{E}[L^0_{U_a}(B)] = a$.

Remark

One can give an alternative proof of the proposition using stochastic calculus. To get (ii), for instance, use Ito's formula to verify that, for every $\lambda > 0$,

$$(1 + \lambda|B_t|)\exp(-\lambda L^0_t(B))$$

is a continuous local martingale, which is bounded on $[0, U_a]$. An application of the optional stopping theorem then shows that $\mathbb{E}[\exp(-\lambda L^0_t(B))] = (1 + \lambda a)^{-1}$.

For every $t \geq 0$, we define

$$S_t := \sup_{s \leq t} B_s, \qquad I_t := \inf_{s \leq t} B_s.$$

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For every $t \geq 0$, we define

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Theorem 9.14 (Lévy)

The two processes $(S_t, S_t - B_t)_{t\geq 0}$ and $(L_t^0(B), |B_t|)_{t\geq 0}$ have the same distribution.

Remark

By an obvious symmetry argument, the pair $(-I_t, B_t - I_t)_{t\geq 0}$ also has the same distribution as $(S_t, S_t - B_t)_{t\geq 0}$.

Proof of Theorem 9.14

By Tanaka's formula, for every $t \geq 0$,

$$|B_t| = -\beta_t + L_t^0(B), \tag{2}$$

where

$$\beta_t = -\int_0^t \mathrm{sgn}(B_s)dB_s.$$

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Proof of Theorem 9.14 (cont)

Since $\langle \beta, \beta \rangle_t = t$, Theorem 5.12 ensures that β is a 1-dim Brownian motion started from 0. We then claim that, for every $t \geq 0$,

$$L_t^0(B) = \sup\{\beta_s : s \leq t\}.$$

The fact that $L_t^0(B) \geq \sup\{\beta_s : s \leq t\}$ is immediate since (2) shows that $L_t^0(B) \geq L_s^0(B) \geq \beta_s$, for every $s \in [0, t]$. To get the reverse inequality, write γ_t for the last zero of B before time t. By the support property of local time, $L_t^0(B) = L_{\gamma_t}^0(B)$ and using (2),
$L_{\gamma_t}^0(B) = \beta_{\gamma_t} \leq \sup\{\beta_s : s \leq t\}$.
We have thus proved a.s.

$$(L_t^0(B), |B_t|)_{t\geq 0} = (\sup\{\beta_s : s \leq t\}, \sup\{\beta_s : s \leq t\} - \beta_t)_{t\geq 0},$$

and since $(\beta_s)_{s\geq 0}$ and $(B_s)_{s\geq 0}$ have the same distribution, the pair in the right-hand side has the same distribution as $(S_t, S_t - B_t)_{t\geq 0}$.

Theorem 9.14 has several interesting consequences. For every $t \geq 0$, S_t has the same law as $|B_t|$ (Theorem 2.21), and thus the same holds for $L_t^0(B)$. From the explicit formula (2.2) for for the density of (S_t, B_t), we also get the density of the pair $(L_t^0(B), B_t)$.

For every $s \geq 0$, define

$$\tau_s := \inf\{t \geq 0 : L_t^0(B) > s\}.$$

The process $(\tau_s)_{s \geq 0}$ is called the inverse local time (at 0) of the Brownian motion B. By construction, $(\tau_s)_{s \geq 0}$ ha cadlag increasing sample paths. From L'evy Theorem 9.14, one gets that

$$(\tau_s)_{s \geq 0} = (\tilde{T}_s)_{s \geq 0}, \qquad \text{in distribution,}$$

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The same application of the strong Markov property as in the proof of Proposition 9.13 shows that $(\tau_s)_{s\geq 0}$ has stationary independent increments. Furthermore, using the invariance of Brownian motion under scaling, we have for every $\lambda > 0$,

$$(\tau_{\lambda s})_{s\geq 0} = (\lambda^2 \tau_s)_{s\geq 0}, \quad \text{in distribution.}$$

The preceding properties can be summarized by saying that $(\tau_s)_{s\geq 0}$ is a stable subordinator with index $1/2$ (a subordinator is a Lévy process with non-decreasing sample paths).

The interest of considering the process $(\tau_s)_{s\geq 0}$ comes in part from the following proposition.

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Proposition 9.15

We have a.s.

$$\{t \geq 0 : B_t = 0\} = \{\tau_s : s \geq 0\} \cup \{\tau_{s-} : s \in D\},$$

where D is the countable set of jump times of $(\tau_s)_{s\geq 0}$.

Proof

We know from (1) that a.s.

$$\mathrm{supp}(d_t L_t^0(B)) \subset \{t \geq 0 : B_t = 0\}.$$

It follows that any time t of the form $t = \tau_s$ or $t = \tau_{s-}$ must belong to the zero set of B. Conversely, recalling that (1) is an a.s. equality for $a = 0$, we also get that, a.s. for every t such that $B_t = 0$, we have either $L^0_{t+\epsilon} > L^0_t(B)$ for every $\epsilon > 0$, or, if $t > 0$, $L^0_{t-\epsilon} < L^0_t(B)$ for every $\epsilon > 0$ with $\epsilon < t$ (or both simultaneously), which implies that we have $t = \tau_{L^0_t(B)}$ or $t = \tau_{L^0_t(B)-}$.

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As a consequence of Proposition 9.15, the connected components of the complement of the zero set $\{t \geq 0 : B_t = 0\}$ are exactly the intervals (τ_{s-}, τ_s) for $s \in D$. These connected components are called the excursion intervals (away from 0). For every $s \in D$, the associated excursion is defined by

$$e_s(t) := B_{(\tau_{s-}+t)\wedge\tau_s}, \quad t \geq 0.$$

The goal of excursion theory is to describe the distribution of the excursion process, that is, of the collection $(e_s)_{s\in D}$. This theory is very useful.