Math 562 Fall 2020

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Outline

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HW7 is due Friday, 12/11, at noon. Please submit your homework via the course Moodle page. Make sure that your HW is uploaded successfully.

HW6 has been graded now.

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Theorem 9.6 (Generalized Ito Formula)

Let f be the difference of convex functions on $\mathbb{R}$. Then, for every $t \geq 0$,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_{\mathbb{R}} L_t^a(X) f''(da).$$

The following corollary is even more important than the preceding theorem.

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Corollary 9.7 (Density of occupation time formula)

We have almost surely, for every $t \geq 0$ and every non-negative measurable function ϕ on $\mathbb{R}$,

$$\int_0^t \phi(X_s) d\langle X, X\rangle_s = \int_{\mathbb{R}} \phi(a) L_t^a(X) da.$$

More generally, we have a.s. for any non-negative measurable function F on $\mathbb{R}_+ \times \mathbb{R}$,

$$\int_0^\infty F(s, X_s) d\langle X, X\rangle_s = \int_{\mathbb{R}} da \int_0^\infty F(s, a) d_s L_s^a(X).$$

Proof

Fix $t \geq 0$ and consider a non-negative continuous function ϕ on $\mathbb{R}$ with compact support. Let f be a twice continuously differentiable function on $\mathbb{R}$ such that $f'' = \phi$. Note that f is convex since $\phi \geq 0$.

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Proof of Corollary 9.7 (cont)

By comparing Ito's formula applied to $f(X_t)$ and the formula of Theorem 9.6, we immediately get that a.s.

$$\int_0^t \phi(X_s)d\langle X, X\rangle_s = \int_{\mathbb{R}} \phi(a)L_t^a(X)da.$$

This formula holds simultaneously (outside a set of probability zero) for every $t \geq 0$ (by a continuity argument) and for every function ϕ belonging to a countable dense subset of the set of all non-negative continuous functions on $\mathbb{R}$ with compact support. This suffices to conclude that a.s. for every $t \geq 0$, the random measure

$$A \mapsto \int_0^t 1_A(X_s)d\langle X, X\rangle_s$$

has density $(L_t^a(X))_{a\in\mathbb{R}}$ with respect to Lebesgue measure on $\mathbb{R}$.

Proof of Corollary 9.7 (cont)

This gives the first assertion of the corollary. It follows that the formula in the second assertion holds when F is of the type

$$F(s,a) = 1_{[u,v]}(s)1_A(a)$$

where $0 \leq u \leq v$ and A is a Borel subset of $\mathbb{R}$. Hence, a.s. the σ-finite measures

$$B \to \int_0^\infty 1_B(s, X_s)d\langle X, X\rangle_s$$

and

$$B \to \int_{\mathbb{R}} da \int_0^\infty 1_B(s,a)d_sL_s^a(X)$$

take the same value for B of the form $B = [u,v] \times A$ and this implies that the two measures coincide.

If $X = M + V$ is a continuous semimartingale, then an immediate application of the density of occupation time formula gives, for every $b \in \mathbb{R}$,

$$\int_0^t 1_{\{X_s=b\}} d\langle M, M\rangle_s = \int_{\mathbb{R}} 1_{\{b\}}(a) L_t^a(X) da = 0.$$

This property has already been derived after the proof of Lemma 9.5. On the other hand, there may exist values of b such that

$$\int_0^t 1_{\{X_s=b\}} dV_s \neq 0,$$

and these values of b correspond to discontinuities of the local time with respect to the space variable, as shown by Theorem 9.4.

Corollary 9.8

If X is of the form $X = X_0 + V$, where V is a finite variation process, then $L_t^a(X) = 0$ for all $a \in \mathbb{R}$ and $t \geq 0$.

Proof

From the density of occupation time formula and the fact that $\langle X, X\rangle_t = 0$, we get $\int_{\mathbb{R}} \phi(a) L_t^a(X) da = 0$ for any non-negative measurable function ϕ, and the desired result follows.

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Proposition 9.9

Let X be a continuous semimartingale. Then a.s. for every $a \in \mathbb{R}$ and $t \geq 0$,

$$L_t^a(X) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^t 1_{\{a \leq X_s \leq a+\epsilon\}} d\langle X, X\rangle_s.$$

Proof

By the density of occupation time formula,

$$\frac{1}{\epsilon} \int_0^t 1_{\{a \leq X_s \leq a+\epsilon\}} d\langle X, X\rangle_s = \frac{1}{\epsilon} \int_a^{a+\epsilon} L_t^b(X) db,$$

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Remark

The same argument gives

$$\lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{a-\epsilon \le X_s \le a+\epsilon\}} d\langle X, X\rangle_s = \frac{1}{2}(L_t^a(X) + L_t^{a-}(X)).$$

The quantity $\tilde{L}_t^a(X) = \frac{1}{2}(L_t^a(X) + L_t^{a-}(X))$ is sometimes called the symmetric local time of the semimartingale X. Note that the density of occupation time formula remains true if $L_t^a(X)$ is replaced by $\tilde{L}_t^a(X)$ (indeed, $\tilde{L}_t^a(X)$ and $L_t^a(X)$ may differ in at most countably many values of a). The generalized Ito formula (Theorem 9.6) also remains true if $L_t^a(X)$ is replaced by $\tilde{L}_t^a(X)$ provided the left derivative f'_- is replaced by $\frac{1}{2}(f'_+ + f'_-)$. Similar observations apply to Tanaka's formulas.

As a consequence of the preceding proposition and Lemma 9.5, we derive a useful bound on moments of local times.

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As a consequence of the preceding proposition and Lemma 9.5, we derive a useful bound on moments of local times.

Corollary 9.10

Let $p \geq 1$. There exists a constant C_p such that, for any continuous semimartingale X with canonical decomposition $X = M + V$, we have for every $a \in \mathbb{R}$ and $t \geq 0$,

$$\mathbb{E}[(L_t^a(X))^p] \leq C_p \left(\mathbb{E}[(\langle M, M \rangle_t)^{p/2}] + \mathbb{E}\left[\left(\int_0^t d|V_s| \right)^p \right] \right).$$

Proof

This readily follows from the bound of Lemma 9.5, using the approximation of $L_t^a(X)$ in Proposition 9.9 and Fatou's lemma.

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We next turn to the upcrossing approximation of local time. We first need to introduce some notation. We let X be a continuous semimartingale, and $\epsilon > 0$. We then introduce two sequences $(\sigma_n^\epsilon)_{n\geq 1}$ and $(\tau_n^\epsilon)_{n\geq 1}$ of stopping times, which are defined inductively by

$$\sigma_1^\epsilon = \inf\{t \geq 0 : X_t = 0\}, \quad \tau_1^\epsilon = \inf\{t \geq \sigma_1^\epsilon : X_t = \epsilon\},$$

and, for every $n \geq 1$,

$$\sigma_{n+1}^\epsilon = \inf\{t \geq \tau_n^\epsilon : X_t = 0\}, \quad \tau_{n+1}^\epsilon = \inf\{t > \sigma_n^\epsilon : X_t = \epsilon\}.$$

We then define the upcrossing number of X along $[0, \epsilon]$ before time t by

$$N_\epsilon^X(t) = \mathrm{Card}\{n \geq 1 : \tau_n^\epsilon \leq t\}.$$

Proposition 9.11

We have, for every $t \geq 0$, as $\epsilon \to 0$,

$$\epsilon N_{\epsilon}^{X}(t) \to \frac{1}{2} L_t^0(X)$$

in probability.

Proof

To simplify notation, we write L_s^0 for $L_s^0(X)$ in this proof. We first use Tanaka's formula to get, for every $n \geq 1$,

$$(X_{\tau_n^\epsilon \wedge t})^+ - (X_{\sigma_n^\epsilon \wedge t})^+ = \int_{\sigma_n^\epsilon \wedge t}^{\tau_n^\epsilon \wedge t} 1_{\{X_s > 0\}} dX_s + \frac{1}{2}(L^0_{\tau_n^\epsilon \wedge t} - L^0_{\sigma_n^\epsilon \wedge t}).$$

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Proof of Proposition 9.11 (cont)

We sum the last identity over all $n \geq 1$ to get

$$\sum_{n=1}^{\infty} (X_{\tau_n^\epsilon \wedge t})^+ - (X_{\sigma_n^\epsilon \wedge t})^+$$
$$= \int_0^t \left(\sum_{n=1}^{\infty} 1_{(\sigma_n^\epsilon, \tau_n^\epsilon]} \right) 1_{\{X_s > 0\}} dX_s + \frac{1}{2} \sum_{n=1}^{\infty} (L^0_{\tau_n^\epsilon \wedge t} - L^0_{\sigma_n^\epsilon \wedge t}). \qquad (1)$$

Note that there are only finitely many values of n such that $\tau_n^\epsilon \leq t$, and that the change of order of the series and the stochastic integral is justified by approximating the series with finite sums and using Proposition 5.8 (the required domination is obvious since the integrands are bounded by 1).

Proof of Proposition 9.11 (cont)

Consider the different terms in (1). Since the local time L^0 does not increase on intervals of the type $[\tau_n^\epsilon, \sigma_{n+1}^\epsilon)$ (nor on $[0, \sigma_1^\epsilon)$), we have

$$\sum_{n=1}^{\infty}(L^0_{\tau_n^\epsilon \wedge t} - L^0_{\sigma_n^\epsilon \wedge t}) = \sum_{n=1}^{\infty}(L^0_{\sigma_{n+1}^\epsilon \wedge t} - L^0_{\sigma_n^\epsilon \wedge t}) = L^0_t.$$

Then, noting that $(X_{\tau_n^\epsilon \wedge t})^+ - (X_{\sigma_n^\epsilon \wedge t})^+ = \epsilon$ if $\tau_n^\epsilon \le t$, we have

$$\sum_{n=1}^{\infty}(X_{\tau_n^\epsilon \wedge t})^+ - (X_{\sigma_n^\epsilon \wedge t})^+ = \epsilon N_\epsilon^X(t) + u(\epsilon),$$

where $0 \le u(\epsilon) \le \epsilon$.

Proof of Proposition 9.11 (cont)

From (1) and the last two displays, the result of the proposition will follow if we can verify that

$$\int_0^t \left(\sum_{n=1}^{\infty} 1_{(\sigma_n^\epsilon, \tau_n^\epsilon]} \right) 1_{\{X_s > 0\}} \, dX_s \to 0$$

in probability as $\epsilon \to 0$. This is again a consequence of Proposition 5.8, since

$$0 \le \left(\sum_{n=1}^{\infty} 1_{(\sigma_n^\epsilon, \tau_n^\epsilon]} \right) 1_{\{X_s > 0\}} \le 1_{\{0 < X_s \le \epsilon\}}$$

and $1_{\{0 < X_s \le \epsilon\}} \to 0$ as $\epsilon \to 0$.