Math 562 Fall 2020

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Outline

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HW7 is due Friday, 12/11, at noon. Please submit your homework via the course Moodle page. Make sure that your HW is uploaded successfully.

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We consider a continuous semimartingale X and write $X = M + V$ for its canonical decomposition. It is convenient to write $L^a(X)$ for the random continuous function $(L^a_t(X))_{t\geq 0}$, which we view as a random variable with values in the space $C(\mathbb{R}_+, \mathbb{R})$. As usual, the latter space is equipped with the topology of uniform convergence on every compact set.

Theorem 9.4

The process $(L^a(X) : a \in \mathbb{R})$ with values in $C(\mathbb{R}_+, \mathbb{R})$ has a cadlag modification, which we consider from now on and for which we keep the same notation $(L^a(X) : a \in \mathbb{R})$. Furthermore, if $L^{a-}(X) = (L^{a-}_t(X))_{t\geq 0}$ denotes the left limit of $b \mapsto L^b(X)$ at a, we have for every $t \geq 0$,

$$L^a_t(X) - L^{a-}_t(X) = 2\int_0^t 1_{\{X_s=a\}}\,dV_s. \tag{1}$$

In particular, if X is a continuous local martingale, the process $(L^a_t(X))_{a\in\mathbb{R},t\geq 0}$ has jointly continuous sample paths.

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$$L^a_t(X) - L^{a-}_t(X) = 2\int_0^t 1_{\{X_s = a\}} dV_s. \qquad (1)$$

In particular, if X is a continuous local martingale, the process $(L^a_t(X))_{a\in\mathbb{R}, t\geq 0}$ has jointly continuous sample paths.

Before we prove Theorem 9.4, we first prove the following

Lemma 9.5

Let $p \geq 1$. There exists a constant C_p, depending only on p, such that for every $a, b \in \mathbb{R}$ with $a < b$, we have

$$\mathbb{E}\left[\left(\int_0^t 1_{\{a<X_s\leq b\}} d\langle M, M\rangle_s\right)^p\right] \leq C_p(b-a)^p \left(\mathbb{E}[(\langle M, M\rangle_t)^{p/2}] + \mathbb{E}\left[\left(\int_0^t d|V_s|\right)^p\right]\right).$$

For every $a \in \mathbb{R}$, write $Y^a = (Y_t^a)_{t\geq 0}$ for the random variable with values in $C(\mathbb{R}_+, \mathbb{R})$ defined by

$$Y_t^a = \int_0^t 1_{\{X_s>a\}} dM_s.$$

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Proof of Lemma 9.5

We start with the first assertion. It is enough to prove that the stated bound holds when $a = -u$ and $b = u$ for some $u > 0$ (then take $u = (b-a)/2$ and replace X by $X - (b+a)/2$). Let f be the unique twice continuously differentiable function such that

$$f''(x) = \left(2 - \frac{|x|}{u}\right)^{+},$$

and $f(0) = f'(0) = 0$. Note that we then have $|f'(x)| \leq 2u$ for every $x \in \mathbb{R}$. Since $f'' \geq 0$ and $f''(x) \geq 1$ for $x \in [-u, u]$, we have

$$\int_0^t 1_{\{-u<X_s\leq u\}} d\langle M, M\rangle_s \leq \int_0^t f''(X_s) d\langle M, M\rangle_s. \tag{2}$$

Proof of Lemma 9.5 (cont)

However, by Ito's formula,

$$\frac{1}{2}\int_0^t f''(X_s)d\langle M,M\rangle_s = f(X_t) - f(X_0) - \int_0^t f'(X_s)dX_s. \tag{3}$$

Since $|f'| \le 2u$, we have

$$\begin{aligned}
\mathbb{E}[|f(X_t) - f(X_0)|^p] &\le (2u)^p \mathbb{E}[|X_t - X_0|^p] \\
&\le (2u)^p \mathbb{E}\left[\left(|M_t - M_0| + \int_0^t d|V_s|\right)^p\right] \\
&\le C_p(2u)^p \left(\mathbb{E}[(\langle M,M\rangle_t)^{p/2}] + \mathbb{E}\left[\left(\int_0^t d|V_s|\right)^p\right]\right)
\end{aligned}$$

using the Burkholder-Davis-Gundy inequalities. Here and below, C_p stands for a constant that depends only on p, which may vary from line to line.

Proof of Lemma 9.5 (cont)

Note

$$\int_0^t f'(X_s)dX_s = \int_0^t f'(X_s)dM_s + \int_0^t f'(X_s)dV_s.$$

We have

$$\mathbb{E}\left[\left|\int_0^t f'(X_s)dV_s\right|^p\right] \leq (2u)^p \mathbb{E}\left[\left(\int_0^t |dV_s|\right)^p\right],$$

and, using the Burkholder-Davis-Gundy inequalities once again,

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^t f'(X_s)dM_s\right|^p\right] &\leq C_p \mathbb{E}\left[\left(\int_0^t f'(X_s)^2 d\langle M, M\rangle_s\right)^{p/2}\right] \\ &\leq C_p (2u)^p \mathbb{E}[(\langle M, M\rangle_t)^{p/2}].\end{aligned}$$

The first assertion of the lemma follows by combining the previous bounds, using (2) and (3).

Proof of Lemma 9.5 (cont)

Now we prove the second assertion. We fix $p > 2$. By the Burkholder-Davis-Gundy inequalities, we have for every $a < b$ and every $t \geq 0$,

$$\mathbb{E}\left[\sup_{s\leq t}|Y_s^b - Y_s^a|^p\right] \leq C_p\mathbb{E}\left[\left(\int_0^t 1_{\{a<X_s\leq b\}}d\langle M,M\rangle_s\right)^{p/2}\right], \quad (4)$$

and the right-hand side can be estimated from the first assertion of the lemma. More precisely, for every integer $n \geq 1$, define the stopping time

$$T_n = \inf\{t \geq 0 : \langle M,M\rangle_t + \int_0^t d|V_s| \geq n\}.$$

From the first assertion of the lemma with X replaced by the stopped process X^{T_n}, we have, for every $t \geq 0$,

Proof of Lemma 9.5 (cont)

$$\mathbb{E}\left[\left(\int_0^{t\wedge T_n} 1_{\{a<X_s\leq b\}}\, d\langle M, M\rangle_s\right)^{p/2}\right] \leq C_p(n^{p/4}+n^{p/2})(b-a)^{p/2}.$$

Using (4), again with X replaced by X^{T_n}, we obtain

$$\mathbb{E}\left[\sup_{s\leq t}\left|Y^b_{s\wedge T_n} - Y^a_{s\wedge T_n}\right|^p\right] \leq C_p(n^{p/4}+n^{p/2})(b-a)^{p/2}.$$

Since $p > 2$, we see that we can apply Kolmogorov's continuity theorem to get the existence of a continuous modification of the process $a \mapsto (Y^a_{s\wedge T_n})_{s\geq 0}$, with values in $C(\mathbb{R}_+, \mathbb{R})$. Write $(Y^{(n),a}_s)_{s\geq 0}$ for this continuous modification.

Proof of Lemma 9.5 (cont)

Then, if $1 \leq n < m$, for every fixed a, we have $Y_s^{(n),a} = Y_{s \wedge T_n}^{(m),a}$ for every $s \geq 0$ a.s. By a continuity argument, the latter equality holds simultaneously for every $a \in \mathbb{R}$ and every $s \geq 0$, outside a single set of probability zero. It follows that we can define a process $(\tilde{Y}^a : a \in \mathbb{R})$ with values in $C(\mathbb{R}_+, \mathbb{R})$, with continuous sample paths, such that, for every $n \geq 1$, $Y_s^{(n),a} = \tilde{Y}^a_{s \wedge T_n}$ for every $a \in \mathbb{R}$ and every $s \geq 0$ a.s. The process $(\tilde{Y}^a : a \in \mathbb{R})$ is the desired continuous modification.

Remark

By applying the bound of Lemma 9.5 to X^{T_n}, and letting a tend to b, we get that, for every $b \in R$,

$$\int_0^t 1_{\{X_s=b\}} d\langle M, M\rangle_s = 0$$

for every $t \geq 0$ a.s. Consequently, using Proposition 4.12, we also have

$$\int_0^t 1_{\{X_s=b\}} dM_s = 0$$

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Proof of Theorem 9.4

With a slight abuse of notation, we still write $(Y^a : a \in \mathbb{R})$ for the continuous modification obtained in the second assertion of Lemma 9.5. We also let $(Z^a : a \in \mathbb{R})$ be the process with values in $C(\mathbb{R}_+, \mathbb{R})$ defined by

$$Z_t^a = \int_0^t 1_{\{X_s > a\}} dV_s.$$

By Tanaka's formula, we have for every fixed $a \in \mathbb{R}$,

$$L_t^a = 2\left((X_t - a)^+ - (X_0 - a)^+ - Y_t^a - Z_t^a\right), \qquad \text{for every } t \geq 0 \text{ a.s.}$$

The right-hand side of the last display provides the desired cadlag modification. Indeed, the process

$$a \mapsto \left((X_t - a)^+ - (X_0 - a)^+ - Y_t^a\right)_{t \geq 0}$$

has continuous sample paths, and on the other hand the process $a \mapsto Z^a$ has cadlag sample paths: for every $a_0 \in \mathbb{R}$, the

Proof of Theorem 9.4 (cont)

dominated convergence theorem shows that

$$\int_0^t 1_{\{X_s > a\}} dV_s \to \int_0^t 1_{\{X_s > a_0\}} dV_s, \quad \text{as } a \downarrow a_0$$
$$\int_0^t 1_{\{X_s > a\}} dV_s \to \int_0^t 1_{\{X_s \geq a_0\}} dV_s, \quad \text{as } a \uparrow a_0$$

uniformly on every compact time interval. The previous display also shows that the jump $Z^{a_0} - Z^{a_0-}$ is given by

$$Z_t^{a_0} - Z_t^{a_0-} = \int_0^t 1_{\{X_s = a_0\}} dV_s,$$

and this completes the proof of the theorem.

From now on, we only deal with the cadlag modification of local times obtained in Theorem 9.4.

Remark

To illustrate Theorem 9.4, set $W_t = |X_t|$, which is also a semimartingale by Tanaka's formula. By the Tanaka's formula (for the postive part) applied to W_t, we have

$$\begin{aligned} W_t = (W_t)^+ &= |X_0| + \int_0^t 1_{\{|X_s|>0\}}(\mathrm{sgn}(X_s)dX_s + dL_s^0(X)) + \frac{1}{2}L_t^0(W) \\ &= |X_0| + \int_0^t \mathrm{sgn}(X_s)dX_s + \int_0^t 1_{\{X_s=0\}}dX_s + \frac{1}{2}L_t^0(W) \end{aligned}$$

noting that $\int_0^t 1_{\{|X_s|>0\}}dL_s^0(X) = 0$ by the support property of local time. Comparing the resulting formula with Tanaka's formula (for abs. value and $a = 0$), we get

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$$L_t^0(W) = 2L_t^0(X) - 2\int_0^t 1_{\{X_s=0\}} dX_s = L_t^0(X) + L_t^{0-}(X),$$

by using (1). The formula $L_t^0(W) = L_t^0(X) + L_t^{0-}(X)$ is a special case of the more general formula $L_t^a(W) = L_t^a(X) + L_t^{-a-}(X)$, for every $a \geq 0$, which is easily deduced from Corollary 9.7 below. We note that the support property of local time implies $L_t^a(W) = 0$ for every $a < 0$, and in particular $L_t^{0-}(W) = 0$.

We will now give an extension of Ito's formula. If f is a convex function on $\mathbb{R}$, the left derivative f'_- is a left-continuous monotone non-decreasing function, and there exists a unique Radon measure $f''(dy)$ on $\mathbb{R}$ such that $f''([a,b)) = f'_-(b) - f'_-(a)$ for every $a < b$. One can also interpret f'' as the second derivative of f in the sense of distributions. Note that $f''(da) = f''(a)da$ if f is twice continuously differentiable. If f is now a difference of convex functions, that is, $f = f_1 - f_2$ where both f_1 and f_2 are convex, we can still make sense of $\int f''(dy)\phi(y) = \int f_1''(dy)\phi(y) - \int f_2''(dy)\phi(y)$ for any bounded measurable function ϕ supported on a compact interval of $\mathbb{R}$.

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Theorem 9.6 (Generalized Ito Formula

Let f be the difference of convex functions on $\mathbb{R}$. Then, for every $t \geq 0$,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_{\mathbb{R}} L_t^a(X) f''(da).$$

Remark

By Proposition 9.2 and a continuity argument, we have

$$L_t^a(X) = 0 \quad \text{for every } a \notin \left[\min_{0\leq s\leq t} X_s, \max_{0\leq s\leq t} X_s\right], \quad a.s.$$

and furthermore the function $a \mapsto L_t^a(X)$ is bounded. Together with the observations preceding the statement of the theorem, this shows that the integral $\int_{\mathbb{R}} L_t^a(X) f''(da)$ makes sense.

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Proof of Theorem 9.6

By linearity, it suffices to treat the case when f is convex. Furthermore, by simple “localization” arguments,we can assume that f'' is a finite measure supported on the interval $[-K, K]$ for some $K > 0$. By adding an affine function to f , we can also assume that $f = 0$ on $(-\infty, -K]$. Then, it is elementary to verify that, for every $x \in \mathbb{R}$,

$$f(x) = \int_{\mathbb{R}} (x - a)^+ f''(da)$$

and

$$f'_-(x) = \int_{\mathbb{R}} 1_{\{a<x\}} f''(da). \tag{5}$$

Tanaka’s formula gives, for every $a \in \mathbb{R}$,

$$(X_t - a)^+ = (X_0 - a)^+ + Y_t^a + Z_t^a + \frac{1}{2} L_t^a(X),$$

where we use the notation of the proof of Theorem 9.4 (and we recall that $Y^a : a \in \mathbb{R})$ stands for the continuous modification obtained in Lemma 9.5).

Proof of Theorem 9.6 (cont)

We can integrate the latter equality with respect to the finite measure $f''(da)$ and we get

$$f(X_t) = f(X_0) + \int Y_t^a f''(da) + \int Z_t^a f''(da) + \frac{1}{2}\int L_t^a(X) f''(da).$$

By Fubini's theorem,

$$\int Z_t^a f''(da) = \int \left(\int_0^t 1_{\{X_s > a\}} dV_s \right) f''(da)$$
$$= \int_0^t \left(\int 1_{\{X_s > a\}} f''(da) \right) dV_s = \int_0^t f'_-(X_s) dV_s.$$

So the proof will be complete if we can also verify that

$$\int Y_t^a f''(da) = \int_0^t f'_-(X_s) dM_s. \tag{6}$$

Proof of Theorem 9.6 (cont)

This identity should be viewed as a kind of Fubini theorem involving a stochastic integral. To provide a rigorous justification, it is convenient to introduce the stopping times $T_n = \inf\{t \geq 0 : \langle M, M \rangle_t \geq n\}$ for every $n \geq 1$. By (5), we see that our claim (6) will follow if we can verify that, for every $n \geq 1$, we have a.s.

$$\int \left(\int_0^{t \wedge T_n} 1_{\{X_s > a\}} dM_s \right) f''(da) = \int_0^{t \wedge T_n} \left(\int 1_{\{X_s > a\}} f''(da) \right) dM_s, \tag{7}$$

where in the left-hand side we use the continuous modification of $a \mapsto \int_0^{t \wedge T_n} 1_{\{X_s > a\}} dM_s$ provided by Lemma 9.5. It is straightforward to verify that the left-hand side of (7) defines a martingale M_t^f in $\mathbb{H}^2$, and furthermore, for any other martingale $N \in \mathbb{H}^2$,

Proof of Theorem 9.6 (cont)

$$
\begin{aligned}
\mathbb{E}[\langle M^f, N\rangle_\infty] &= \mathbb{E}[M^f_\infty N_\infty] \\
&= \mathbb{E}\left[\int \left(\int_0^{T_n} 1_{\{X_s > a\}} d\langle M, N\rangle_s\right) f''(da)\right] \\
&= \mathbb{E}\left[\int_0^{T_n} \left(\int 1_{\{X_s > a\}} f''(da)\right) d\langle M, N\rangle_s\right] \\
&= \mathbb{E}\left[\left(\int_0^{T_n} \left(\int 1_{\{X_s > a\}} f''(da)\right) dM_s\right) N_\infty\right].
\end{aligned}
$$

By a duality argument in $\mathbb{H}^2$, this suffices to verify that M^f_t coincides with the martingale of $\mathbb{H}^2$ in the right-hand side of (7). This completes the proof.