Math 562 Fall 2020

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December 02, 2020

Outline

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1. General Info
2. 8.4 A Few Examples of Stochastic Differential Equations
3. 9.1 Tanaka's Formula and the Definition of Local Times
4. 9.2 Continuity of Local Times and the Generalized Ito Formula

HW7 is due Friday, 12/11, at noon. Please submit your homework via the course Moodle page. Make sure that your HW is uploaded successfully.

Outline

8.4.3 Bessel Processes

Let $m \geq 0$. The m-dimensional squared Bessel process is the real-valued process taking non-negative values that solves the stochastic differential equation

$$dX_t = 2\sqrt{X_t}dB_t + mdt. \tag{1}$$

Note that this equation does not fit into the Lipschitz setting studied in this chapter, because the function $\sigma(x) = 2\sqrt{x}$ is not Lipschitz on $\mathbb{R}_+$. However, there exist (especially in dimension one) criteria weaker than our Lipschitz continuity assumptions, which apply to (1) and give the existence and pathwise uniqueness of solutions of (1). For instance, see Exercise 8.14 for a criterion of pathwise uniqueness that applies to (1).

One of the main reasons for studying Bessel processes comes from the following observation. If $d \geq 1$ is an integer and $\beta = (\beta^1, \ldots, \beta^d)$ is a d-dimensional Brownian motion, an application of Ito's formula shows that the process

$$|\beta_t|^2 = (\beta_t^1)^2 + \cdots + (\beta_t^d)^2$$

is a d-dimensional squared Bessel process.

Suppose from now on that $m > 0$ and $X_0 = x > 0$. For every $r \geq 0$, define $T_r = \inf\{t \geq 0 : X_t = r\}$. If $r > x$, we have $\mathbb{P}(T_r < \infty) = 1$. To show this, use (1) to see that $X_{t \wedge T_r} = x + m(t \wedge T_r) + Y_{t \wedge T_r}$, where $\mathbb{E}[(Y_{t \wedge T_r})^2] \leq 4rt$. By Markov's inequality, $\mathbb{P}(Y_{t \wedge T_r} > t^{3/4}) \to 0$ as $t \to \infty$, and if we assume that $\mathbb{P}(T_r = \infty) > 0$ the preceding expression for $X_{t \wedge T_r}$ gives a contradiction.

Define, for every $t \in [0, T_0)$,

$$M_t = \begin{cases} (X_t)^{1-\frac{m}{2}}, & m \neq 2, \\ \log(X_t), & , m = 2. \end{cases}$$

It follows from Ito's formula that, for every $\epsilon \in (0, x)$, $M_{t \wedge T_\epsilon}$ is a continuous local martingale. This continuous local martingale is bounded over the time interval $[0, T_\epsilon \wedge T_A]$, for every $A > x$, and an application of the optional stopping theorem (using the fact that $T_A < \infty$ a.s.) gives, if $m \neq 2$,

$$\mathbb{P}(T_\epsilon < T_A) = \frac{A^{1-\frac{m}{2}} - x^{1-\frac{m}{2}}}{A^{1-\frac{m}{2}} - \epsilon^{1-\frac{m}{2}}},$$

and if $m = 2$,

$$\mathbb{P}(T_\epsilon < T_A) = \frac{\log A - \log x}{\log A - \log \epsilon}.$$

Let's finally concentrate on the case $m \geq 2$. Let $\epsilon \downarrow 0$ in the preceding formulas, we obtain that $\mathbb{P}(T_0 < \infty) = 0$. If we let $A \uparrow \infty$, we also get that $\mathbb{P}(T_\epsilon < \infty) = 1$ if $m = 2$ and $\mathbb{P}(T_\epsilon < \infty) = (\epsilon/x)^{(m/2)-1}$ if $m > 2$.

It then follows from the property $\mathbb{P}(T_0 < \infty) = 0$ that the process M_t is well defined for every $t \geq 0$ and is a continuous local martingale. When $m > 2$, M_t takes non-negative values and is thus a supermartingale, which converges a.s. as $t \to \infty$. The limit must be 0, since we already noticed that $\mathbb{P}(T_A < \infty) = 1$ for every $A > x$ and we conclude that X_t converges a.s. to ∞ as $t \to \infty$ when $m > 2$. One can show that the continuous local martingale M_t is not a (true) martingale.

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In this chapter, we apply stochastic calculus to the theory of local times of continuous semimartingales. Roughly speaking, the local time at level a of a semimartingale X is an increasing process that measures the "time" that X spent at level a.

Throughout this chapter, we work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, and the filtration $(\mathcal{F}_t)$ is assumed to be complete. Let X be a continuous semimartingale. If f is a twice continuously differentiable function defined on $\mathbb{R}$, Ito's formula asserts that $f(X_t)$ is still a continuous semimartingale, and

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle X, X\rangle_s.$$

The next proposition shows that this formula can be extended to the case when f is a convex function.

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The next proposition shows that this formula can be extended to the case when f is a convex function.

Proposition 9.1

Let f be a convex function on $\mathbb{R}$. Then $f(X_t)$ is a semimartingale, and, more precisely, there exists an increasing process A^f such that, for every $t \geq 0$,

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)dX_s + A_t^f$$

where $f'_-(x)$ denotes the left-derivative of f at x.

More generally, $f(X_t)$ is a semimartingale if f is a difference of convex functions.

Proof

Let h be a non-negative continuous function on $\mathbb{R}$ such that $h(x) = 0$ for $x \notin [0, 1]$ and $\int_0^1 h(x)dx = 1$. For every integer $n \geq 1$, define $h_n(x) = nh(nx)$. Define a function $\phi_n : \mathbb{R} \to \mathbb{R}$ by

$$\phi_n(x) = h_n * f(x) = \int_{-\infty}^{\infty} h_n(y)f(x - y)dy.$$

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Proof of Proposition 9.1

Then it is elementary to verify that ϕ_n is twice continuously differentiable on $\mathbb{R}$, $\phi_n' = h_n * f_-'$, $\phi_n(x) \to f(x)$ and $\phi_n'(x) \to f_-'(x)$ for every $x \in \mathbb{R}$ as $n \to \infty$. Furthermore, the functions ϕ_n are also convex, so that $\phi_n'' \geq 0$.

Let $X = M + V$ be the canonical decomposition of the semimartingale X, and consider an integer $K \geq 1$. Introduce the stopping time

$$T_K = \inf\{t \geq 0 : |X_t| + \langle M, M \rangle_t + \int_0^t |dV_s| \geq K\}.$$

By Ito's formula, we have

$$\phi_n(X_{t\wedge T_K}) = \phi_n(X_0) + \int_0^{t\wedge T_K} \phi_n'(X_s)dX_s + \frac{1}{2}\int_0^{t\wedge T_K} \phi_n''(X_s)d\langle M, M \rangle_s. \tag{2}$$

Proof of Proposition 9.1 (cont)

From the definition of T_K, we have $\langle M, M\rangle_{T_K} \leq K$. Noting that the functions ϕ_n' are uniformly bounded over any compact interval, we get, by a simple application of Proposition 5.8, as $n \to \infty$,

$$\int_0^{t\wedge T_K} \phi_n'(X_s)dX_s \to \int_0^{t\wedge T_K} f_-'(X_s)dX_s \tag{3}$$

in probability. For every $t \geq 0$, define

$$A_t^{f,K} = f(X_{t\wedge T_K}) - f(X_0) - \int_0^{t\wedge T_K} f_-'(X_s)dX_s. \tag{4}$$

Since $\phi_n(X_0) \to f(X_0)$ and $\phi_n(X_{t\wedge T_K}) \to f(X_{t\wedge T_K})$ as $n \to \infty$, we deduce from (3) and (2) that

$$\frac{1}{2}\int_0^{t\wedge T_K} \phi_n''(X_s)d\langle M, M\rangle_s \to A_t^{f,K} \tag{5}$$

in probability.

Proof of Proposition 9.1 (cont)

By (4), the process $(A_t^{f,K})_{t\geq 0}$ has continuous sample paths, and $A_0^{f,K} = 0$. Since $\phi_n'' \geq 0$, it follows from the convergence (5) that the sample paths of $(A_t^{f,K})_{t\geq 0}$ are also non-decreasing. Hence $A^{f,K}$ is an increasing process. Finally, one gets from (5) that $A_t^{f,K} = A_{t\wedge T_K}^{f,K'}$ if $K \leq K'$. It follows that there exists an increasing process A^f such that $A_t^{f,K} = A_{t\wedge T_K}^f$ for every $t \geq 0$ and $K \geq 1$. We then get the formula of the proposition by letting $K \uparrow \infty$ in (4).

Remark

Write f'_+ for the right-derivative of f . An argument similar to the preceding proof shows that there exists an increasing process $\tilde{A}^f$ such that

$$f(X_t) = f(X_0) + \int_0^t f'_+(X_s)dX_s + \tilde{A}^f_t.$$

If f is twice continuously differentiable, $A^f_t = \tilde{A}^f_t = \frac{1}{2}\int_0^t f''(X_s)d\langle X, X\rangle_s$. In general, however, we may have $A^f_t \neq \tilde{A}^f_t$.

The previous proposition leads to an easy definition of the local times of a semimartingale. For every $x \in \mathbb{R}$, we define $\mathrm{sgn}(x) = 1_{\{x>0\}} - 1_{\{x\leq 0\}}$ (the fact that we define $\mathrm{sgn}(0) = -1$ here plays a significant role).

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Proposition 9.2

Let X be a continuous semimartingale and $a \in \mathbb{R}$. There exists an increasing process $(L_t^a(X))_{t\geq 0}$ such that the following three identities hold:

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a) dX_s + L_t^a(X) \tag{6}$$

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t 1_{\{X_s > a\}} dX_s + \frac{1}{2} L_t^a(X) \tag{7}$$

$$(X_t - a)^- = (X_0 - a)^- - \int_0^t 1_{\{X_s \leq a\}} dX_s + \frac{1}{2} L_t^a(X). \tag{8}$$

The increasing process $(L_t^a(X))_{t\geq 0}$ is called the local time of X at level a. Furthermore, for every stopping time T, we have $L_t^a(X^T) = L_{t\wedge T}^a(X)$.

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Proof of Proposition 9.2

We apply Proposition 9.1 to the convex function $f(x) = |x - a|$, noting that $f'_-(x) = \mathrm{sgn}(x - a)$. It follows from Proposition 9.1 that the process $(L^a_t(X))_{t\geq 0}$ defined by

$$L^a_t(X) = |X_t - a| - |X_0 - a| - \int_0^t \mathrm{sgn}(X_s - a)dX_s$$

is an increasing process. We then need to verify that (7) abd (8) hold. To this end, we apply Proposition 9.1 to the convex functions $f(x) = (x - a)^+$ and $f(x) = (x - a)^-$. It follows that there exist two increasing processes $A^{a,(+)}$ and $A^{a,(-)}$ such that

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t 1_{\{X_s > a\}} dX_s + A^{a,(+)}_t$$

and

$$(X_t - a)^- = (X_0 - a)^- - \int_0^t 1_{\{X_s \leq a\}} dX_s + A^{a,(-)}_t.$$

Proof of Proposition 9.2 (cont)

By considering the difference between the last two displays, we immediately get that $A^{a,(+)} = A^{a,(-)}$. On the other hand, if we add these two displays and compare with (6), we get $A_t^{a,(+)} + A_t^{a,(-)} = L_t^a(X)$. Hence $A_t^{a,(+)} = A_t^{a,(-)} = \frac{1}{2} L_t^a(X)$.

The last assertion immediately follows from (6) since

$$\int_0^{t \wedge T} \operatorname{sgn}(X_s - a) dX_s = \int_0^t \operatorname{sgn}(X_s^T - a) dX_s^T$$

by properties of the stochastic integral.

Let us state the key property of local times. We use the notation $d_s L_s^a(X)$ for the random measure associated with the increasing function $s \mapsto L_s^a(X)$.

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Proposition 9.3

Let X be a continuous semimartingale and let $a \in \mathbb{R}$. Then a.s. the random measure $d_s L_s^a(X)$ is supported on $\{s \geq 0 : X_s = a\}$.

Proof

Define $W = |X_t - a|$ and note that (6) gives $\langle W, W \rangle_t = \langle X, X \rangle_t$ since $|\mathrm{sgn}(x)| = 1$ for every $x \in \mathbb{R}$. By applying Ito's formula to $(W_t)^2$, we get

$$(X_t - a)^2 = (X_0 - a)^2 + 2\int_0^t (X_s - a) dX_s + 2\int_0^t |X_s - a| d_s L_s^a(X) + \langle X, X \rangle_t.$$

Comparing with the result of a direct application of Ito's formula to $(X_t - a)^2$, we get

$$\int_0^t |X_s - a| d_s L_s^a(X) = 0,$$

which gives the desired result.

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Proposition 9.3 shows that the function $t \mapsto L_t^a(X)$ may only increase when $X_t = a$. So in some sense, $L_t^a(X)$ measures the "time" spent by the process X at level a before time t. This also justifies the name "local time".

In the next section, we study the continuity of the local times of X with respect to the space variable a.

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We consider a continuous semimartingale X and write $X = M + V$ for its canonical decomposition. It is convenient to write $L^a(X)$ for the random continuous function $(L^a_t(X))_{t\geq 0}$, which we view as a random variable with values in the space $C(\mathbb{R}_+, \mathbb{R})$. As usual, the latter space is equipped with the topology of uniform convergence on every compact set.

Theorem 9.4

The process $(L^a(X) : a \in \mathbb{R})$ with values in $C(\mathbb{R}_+, \mathbb{R})$ has a cadlag modification, which we consider from now on and for which we keep the same notation $(L^a(X) : a \in \mathbb{R})$. Furthermore, if $L^{a-}(X) = (L^{a-}_t(X))_{t\geq 0}$ denotes the left limit of $b \mapsto L^b(X)$ at a, we have for every $t \geq 0$,

$$L^a_t(X) - L^{a-}_t(X) = 2\int_0^t 1_{\{X_s=a\}}\, dV_s. \tag{9}$$

In particular, if X is a continuous local martingale, the process $(L^a_t(X))_{a\in\mathbb{R},t\geq 0}$ has jointly continuous sample paths.

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