Math 562 Fall 2020

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Outline

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HW6 is due today at noon. Please submit your homework via the course Moodle page. Make sure that your HW is uploaded successfully.

HW5 has been graded now.

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Theorem 8.5

Under the Lipschitz assumption, there exists, for every $x \in \mathbb{R}^d$, a mapping $F_x : C(\mathbb{R}_+, \mathbb{R}^m) \to C(\mathbb{R}_+, \mathbb{R}^d)$, which is measurable when $C(\mathbb{R}_+, \mathbb{R}^m)$ is equipped with the Borel σ-field completed by the W-negligible sets, and $C(\mathbb{R}_+, \mathbb{R}^d)$ is equipped with the Borel σ-field, such that the following properties hold:

- **(i)** for every $t \geq 0$, $F_x(w)_t$ coincides $W(dw)$ a.s. with a measurable function of $(w_r : r \in [0, t])$;
- **(ii)** For every $w \in C(\mathbb{R}_+, \mathbb{R}^m)$, the mapping $x \mapsto F_x(w)$ is continuous;
- **(iii)** for every $x \in \mathbb{R}^d$, for every choice of the (complete) filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and of the m-dimensional $(\mathcal{F}_t)$-Brownian motion B with $B_0 = 0$, the process X_t defined $X_t = F_x(B)_t$ is the unique solution of $E_x(\sigma, b)$; furthermore, if U is an $\mathcal{F}_0$-measurable real-valued random variable, the process $F_U(B)_t$ is the unique solution with $X_0 = U$.

Proof of Theorem 8.5 (cont)

Last time we have proved parts (i) and (ii). Now we prove the first part of assertion (iii). To this end, we fix the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and the $(\mathcal{F}_t)$-Brownian motion B. We need to verify that the process $(F_x(B)_t)_{t\geq 0}$ then solves $E_x(\sigma, b)$. This process (trivially) has continuous sample paths, and is also adapted since $F_x(B)_t$ coincides a.s. with a measurable function of $(B_r : 0 \leq r \leq t)$, by (i), and since the filtration $(\mathcal{F}_t)$ is complete. On the other hand, by the construction of F_x, (and because $\widetilde{X}^x = X^x$ a.s.), we have, for every $t \geq 0$, $W(dw)$ a.s.,

$$F_x(w)_t = x + \int_0^t \sigma(s, F_x(w)_s)dw(s) + \int_0^t b(s, F_x(w)_s)ds,$$

where the stochastic integral $\int_0^t \sigma(s, F_x(w)_s)dw(s)$ can be defined by

Proof of Theorem 8.5 (cont)

$$\int_0^t \sigma(s, F_x(w)_s)dw(s)$$
$$= \lim_{k\to\infty} \sum_{j=0}^{2^{n_k}-1} \sigma(jt/2^{n_k}, F_x(w)_{jt/2^{n_k}}) \left(w\left(\frac{(j+1)t}{2^{n_k}}\right) - w\left(\frac{jt}{2^{n_k}}\right)\right),$$

$W(dw)$ a.s. Here $(n_k)_{k\geq 1}$ is a suitable subsequence, and we used Proposition 5.9. We can now replace w by B (whose distribution is $W(dw)$) and get a.s. $F_x(B)_t$ is equal to

$$x + \lim_{k\to\infty} \sum_{j=0}^{2^{n_k}-1} \sigma(jt/2^{n_k}, F_x(B)_{jt/2^{n_k}}) \left(B_{(j+1)t/2^{n_k}} - B_{jt/2^{n_k}}\right) + \int_0^t b(s, F_x(B)_s)ds$$
$$= x + \int_0^t \sigma(s, F_x(B)_s)dB(s) + \int_0^t b(s, F_x(B)_s)ds$$

again thanks to Proposition 5.9. We thus obtain that $F_x(B)$ is the desired solution.

Proof of Theorem 8.5 (cont)

We still have to prove the second part of assertion (iii). We again fix the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and the $(\mathcal{F}_t)$-Brownian motion B. Let U be an $\mathcal{F}_0$-measurable random variable. If in the stochastic integral equation satisfied by $F_x(B)$ we formally substitute U for x, we obtain that $F_U(B)$ solves $E(\sigma, b)$ with initial value U. However, this formal substitution is not so easy to justify, and we will argue with some care.

We first observe that the mapping $(x, \omega) \mapsto F_x(B)_t$ is continuous with respect to the variable x (if ω is fixed) and $\mathcal{F}_t$-measurable with respect to ω (if x is fixed). It easily follows that this mapping is measurable for the σ-field $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t$. Since U is $\mathcal{F}_0$-measurable, we get that $F_U(B)_t$ is $\mathcal{F}_t$-measurable. Hence the process $F_U(B)$ is adapted.

Proof of Theorem 8.5 (cont)

For $x \in \mathbb{R}$ and $w \in C(\mathbb{R}_+, \mathbb{R})$, we define $G(x, w) \in C(\mathbb{R}_+, \mathbb{R})$ by the formula

$$G(x, w)_t = \int_0^t b(s, F_x(w)_s) ds.$$

We also set $H(x, w) = F_x(w) - x - G(x, w)$. We have already seen that, for every $x \in \mathbb{R}$, we have $W(dw)$ a.s.

$$H(x, w)_t = \int_0^t \sigma(s, F_x(w)_s) dw(s).$$

Hence, if

$$H_n(x, w)_t = \sum_{j=0}^{2^n-1} \sigma(jt/2^n, F_x(w)_{jt/2^n}) \left(w\left(\frac{(j+1)t}{2^n}\right) - w\left(\frac{jt}{2^n}\right)\right),$$

Proof of Theorem 8.5 (cont)

Proposition 5.9 shows that

$$H(x,w)_t = \lim_{n\to\infty} H_n(x,w)_t$$

in probability under $W(dw)$, for every $x \in \mathbb{R}$. Using the fact that U and B are independent (because U is $\mathcal{F}_0$-measurable), we infer from the latter convergence that

$$H(U,B)_t = \lim_{n\to\infty} H_n(U,B)_t$$

in probability. Thanks again to Proposition 5.9, the limit must be the stochastic integral

$$\int_0^t \sigma(s, F_U(B)_s) dB_s.$$

We have thus proved that

$$\int_0^t \sigma(s, F_U(B)_s) sB_s = H(U,B)_t = F_U(B)_t - U - \int_0^t b(s, F_U(B)_s) ds.$$

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which shows that $F_U(B)$ solves $E(\sigma, b)$ with initial value U.

A consequence of Theorem 8.5, especially of property (ii) in this theorem, is the continuity of solutions with respect to the initial value. Given the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and the $(\mathcal{F}_t)$-Brownian motion B, one can construct, for every $x \in \mathbb{R}^d$, the solution X^x of $E_x(\sigma, b)$ in such a way that, for every $\omega \in \Omega$, the mapping $x \mapsto X^x(\omega)$ is continuous. More precisely, the arguments of

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$$\sup_{t \leq T} |X_t^x(\omega) - X_t^y(\omega)| \leq C_{\epsilon,A,T}(\omega)|x-y|^{1-\epsilon}$$

(in fact the version of Kolmogorov's continuity theorem in Theorem 2.9 gives this only for $d = 1$, but there is an analogous version of Kolmogorov's continuity theorem for processes indexed by a multidimensional parameter.

Outline

In this section, we consider the homogeneous case where $\sigma(t,y) = \sigma(y)$ and $b(t,y) = b(y)$. As in the previous section, we assume that σ and b are Lipschitz: There exists a constant K such that, for every $x, y \in \mathbb{R}^d$,

$$|\sigma(x) - \sigma(y)| \le K|x - y|, \quad |b(x) - b(y)| \le K|x - y|.$$

Let $x \in \mathbb{R}^d$, and let X^x be a solution of $E_x(\sigma, b)$. Since weak uniqueness holds, for every $t \ge 0$, the law of X_t^x does not depend on the choice of the solution. In fact, this law is the image of the Wiener measure on $C(\mathbb{R}_+, \mathbb{R}^d)$ under the mapping $w \mapsto F_x(w)_t$, where the mappings F_x were introduced in Theorem 8.5. The latter theorem also shows that the law of X_t^x depends continuously on the pair (x, t).

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Theorem 8.6

Assume that $(X_t)_{t\geq 0}$ is a solution of $E(\sigma, b)$ on a (complete) filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Then $(X_t)_{t\geq 0}$ is a Markov process with respect to the filtration $(\mathcal{F}_t)$, with semigroup $(Q_t)_{t\geq 0}$ defined by

$$Q_t f(x) = \mathbb{E}[f(X_t^x)],$$

where X^x is an arbitrary solution of $E_x(\sigma, b)$.

With the notation of Theorem 8.5, we have also

$$Q_t f(x) = \int f(F_x(w)_t) W(dw). \tag{1}$$

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Proof of Theorem 8.6

We first verify that, for any bounded measurable function f on $\mathbb{R}^d$, and for every $s, t \geq 0$, we have

$$\mathbb{E}[f(X_{s+t})|\mathcal{F}_s] = Q_t f(X_s).$$

where $Q_t f$ is defined by (1). To this end, we fix $s \geq 0$ and we write, for every $t \geq 0$,

$$X_{s+t} = X_s + \int_s^{s+t} \sigma(X_r) dB_r + \int_s^{s+t} b(X_r) dr, \tag{2}$$

where B is an $(\mathcal{F}_t)$-Brownian motion starting from 0. We then set, for every $t \geq 0$,

$$X'_t = X_{s+t}, \quad \mathcal{F}'_t = \mathcal{F}_{s+t}, \quad B'_t = B_{s+t} - B_s.$$

Proof of Theorem 8.6 (cont)

We observe that the filtration $(\mathcal{F}'_t)$ is complete (of course $\mathcal{F}'_\infty = \mathcal{F}_\infty$), that the process X' is adapted to $(\mathcal{F}'_t)$, and that B' is an m-dimensional $(\mathcal{F}'_t)$-Brownian motion. Furthermore, using the approximation results for the stochastic integral of adapted processes with continuous sample paths (Proposition 5.9), one easily verifies that, a.s. for every $t \geq 0$,

$$\int_s^{s+t} \sigma(X_r)dB_r = \int_0^t \sigma(X'_u)dB'_u$$

where the stochastic integral in the right-hand side is computed in the filtration $(\mathcal{F}'_t)$. It follows from (2) that

$$X'_t = X_s + \int_0^t \sigma(X'_u)dB'_u + \int_0^t b(X'_u)du.$$

Proof of Theorem 8.6 (cont)

Hence X' solves $E(\sigma, b)$, on the space $(\Omega, \mathcal{F}, (\mathcal{F}'_t), \mathbb{P})$ and with the Brownian motion B', with initial value $X'_0 = X_s$ (note that X_s is $\mathcal{F}'_0$-measurable). By the last assertion of Theorem 8.5, we must have $X' = F_{X_s}(B')$, a.s.

Consequently, for every $t \geq o$,

$$\mathbb{E}[f(X_{s+t})|\mathcal{F}_s] = \mathbb{E}[f(X'_t)|\mathcal{F}_s] = \mathbb{E}[f(F_{X_s}(B')_t)|\mathcal{F}_s]$$
$$= \int f(F_{X_s}(w)_t)W(dw) = Q_t f(X_s)$$

by the definition of $Q_t f$. In the third equality, we used the fact that B' is independent of $\mathcal{F}_s$, and distributed according to $W(dw)$, whereas X_s is $\mathcal{F}_s$-measurable.

Proof of Theorem 8.6 (cont)

We still have to verify that $(Q_t)_{t\geq 0}$ is a transition semigroup. Properties (i) and (iii) of the definition are immediate (for (iii), we use the fact that the law of X_t^x depends continuously on the pair (x,t)). For the Chapman-Kolmogorov relation, we observe that, by applying the preceding considerations to X^x, we have, for every $s, t \geq 0$,

$$Q_{s+t}f(x) = \mathbb{E}[f(X_{s+t}^x)] = \mathbb{E}[\mathbb{E}[f(X_{s+t}^x)|\mathcal{F}_s]]$$
$$= \mathbb{E}[Q_t f(X_s^x)] = \int Q_s(x,dy)Q_t f(y).$$

This completes the proof.