Math 562 Fall 2020

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Outline

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HW6 is due Friday, 11/20, at noon. Please submit your homework via the course Moodle page. Make sure that your HW is uploaded successfully.

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Assumptions

The functions σ and b are continuous on $\mathbb{R}_+ \times \mathbb{R}^d$ and Lipschitz in the variable x: There exists a constant K such that, for every $t \geq 0, x, y \in \mathbb{R}^d$,

$$|\sigma(t,x) - \sigma(t,y)| \leq K|x - y|;$$
$$|b(t,x) - b(t,y)| \leq K|x - y|.$$

Theorem 8.3

Under the preceding assumptions, pathwise uniqueness holds for $E(\sigma, b)$, and, for every choice of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty]}, \mathbb{P})$ and of the $(\mathcal{F}_t)$-Brownian motion B, for every $x \in \mathbb{R}^d$, there exists a (unique) strong solution of $E_x(\sigma, b)$.

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Proof of Theorem 8.3

Last time we have proved the first assertion of the theorem, that is, pathwise uniqueness holds. Now we prove the second assertion. We construct a solution using Picard's approximation method.

Define inductively

$$\begin{aligned} X_t^0 &= x, \\ X_t^1 &= x + \int_0^t \sigma(s, x) dB_s + \int_0^t b(s, x) ds, \\ X_t^n &= x + \int_0^t \sigma(s, X_s^{n-1}) dB_s + \int_0^t b(s, X_s^{n-1}) ds. \end{aligned}$$

The stochastic integrals are well defined since one verifies by induction that, for every n, the process X^n is adapted and has continuous sample paths.

Proof of Theorem 8.3 (cont)

It suffices to show that, for every $T > 0$, there is a strong solution of $E_x(\sigma, b)$ on the time interval $[0, T]$. Indeed, the uniqueness part of the argument will then allow us to get a (unique) strong solution on $\mathbb{R}_+$ that will coincide with the solution on $[0, T]$ up to time T.

We fix $T > 0$ and, for every $n \geq 1$ and every $t \in [0, T]$, we set

$$g_n(t) = \mathbb{E}\left[\sup_{s\in[0,t]} |X_s^n - X_s^{n-1}|^2\right].$$

We will bound the functions g_n by induction on n (at present, it is not yet clear that these functions are finite). The fact that the functions $\sigma(\cdot, x)$ and $b(\cdot, x)$ are continuous, hence bounded, over $[0, T]$ implies that there exists a constant C'_T such that $g_1(t) \leq C'_T$ for every $t \in [0, T]$ (use Doob's L^2 inequality for the stochastic integral term).

Proof of Theorem 8.3 (cont)

Then we observe that

$$X_t^{n+1} - X_t^n$$
$$= \int_0^t \left(\sigma(s, X_s^n) - \sigma(s, X_s^{n-1})\right) dB_s + \int_0^t \left(b(s, X_s^n) - b(s, X_s^{n-1})\right) ds.$$

Hence, using the case $p = 2$ of the Burkholder-Davis-Gundy inequalities in the second bound (and writing $C_{(2)}$ for the constant in this inequality),

$$\mathbb{E}\left[\sup_{s\in[0,t]} |X_s^{n+1} - X_s^n|^2\right]$$
$$\leq 2\mathbb{E}\left[\sup_{s\in[0,t]} \left|\int_0^s \left(\sigma(u, X_u^n) - \sigma(u, X_u^{n-1})\right) dB_u\right|^2\right]$$
$$+ 2\mathbb{E}\left[\sup_{s\in[0,t]} \left|\int_0^s \left(b(u, X_u^n) - b(u, X_u^{n-1})\right) du\right|^2\right]$$

Proof of Theorem 8.3 (cont)

$$
\begin{aligned}
&\leq 2C_{(2)}\mathbb{E}\left[\int_0^t \left(\sigma(u,X_u^n)-\sigma(u,X_u^{n-1})\right)^2 du\right] \\
&\quad + 2T\mathbb{E}\left[\int_0^t \left(b(u,X_u^n)-b(u,X_u^{n-1})\right)^2 du\right] \\
&\leq 2(C_{(2)}+T)K^2\mathbb{E}\left[\int_0^t |X_u^n - X_u^{n-1}|^2 du\right] \\
&\leq C_T\mathbb{E}\left[\int_0^t \sup_{r\in[0,u]} |X_r^n - X_r^{n-1}|^2 du\right],
\end{aligned}
$$

where $C_T = 2(C_{(2)}+T)K^2$. We have thus obtained that, for every $n \geq 1$,

$$
g_{n+1}(t) \leq C_T \int_0^t g_n(u)du \tag{1}
$$

Proof of Theorem 8.3 (cont)

Recall that $g_1(t) \le C'_T$. An induction argument using (1) shows that, for every $n \ge 1$ and $t \in [0, T]$,

$$g_n(t) \le C'_T (C_T)^{n-1} \frac{t^{n-1}}{(n-1)!}.$$

In particular, $\sum_{n=1}^{\infty} g_n(T)^{1/2} < \infty$, which implies that

$$\sum_{n=0}^{\infty} \sup_{t \in [0,T]} |X_t^{n+1} - X_t^n| < \infty, \quad a.s.$$

Hence the sequence $(X_t^n : t \in [0, T])$ converges uniformly on $[0, T]$ a.s., to a limiting process $(X_t : t \in [0, T])$, which has continuous sample paths. By induction, one also verifies that, for every n, X^n is adapted with respect to the (completed) canonical filtration of B, and the same holds for X.

Proof of Theorem 8.3 (cont)

Finally, from the fact that σ and b are Lipschitz in the variable x, we also get that, for every $t \in [0, T]$,

$$\lim_{n\to\infty} \left(\int_0^t \sigma(s, X_s) dB_s - \int_0^t \sigma(s, X_s^n) dB_s \right) = 0,$$

$$\lim_{n\to\infty} \left(\int_0^t b(s, X_s) ds - \int_0^t b(s, X_s^n) ds \right) = 0,$$

in probability (to deal with the stochastic integrals, we may use Proposition 5.8, noting that $|X_s^n - X_s|$ is dominated by $\sum_{k=0}^{\infty} \sup_{r\in[0,s]} |X_r^{k+1} - X_r^k|$). By passing to the limit in the induction equation defining X^n, we get that X solves $E_x(\sigma, b)$ on $[0, T]$. This completes the proof of the theorem.

In the following statement, $W(dw)$ stands for the Wiener measure on the canonical space $C(\mathbb{R}_+, \mathbb{R}^m)$ of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^m$. ($W(dw)$ is the law of $(B_t : t \geq 0)$ if B is an m-dimensional Brownian motion started from 0).

Theorem 8.5

Under the assumptions of the preceding theorem, there exists, for every $x \in \mathbb{R}^d$, a mapping $F_x : C(\mathbb{R}_+, \mathbb{R}^m) \to C(\mathbb{R}_+, \mathbb{R}^d)$, which is measurable when $C(\mathbb{R}_+, \mathbb{R}^m)$ is equipped with the Borel σ-field completed by the W-negligible sets, and $C(\mathbb{R}_+, \mathbb{R}^d)$ is equipped with the Borel σ-field, such that the following properties hold:

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Theorem 8.5 (cont)

(i) for every $t \geq 0$, $F_x(w)_t$ coincides $W(dw)$ a.s. with a measurable function of $(w_r : r \in [0, t])$;

(ii) For every $w \in C(\mathbb{R}_+, \mathbb{R}^m)$, the mapping $x \mapsto F_x(w)$ is continuous;

(iii) for every $x \in \mathbb{R}^d$, for every choice of the (complete) filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and of the m-dimensional $(\mathcal{F}_t)$-Brownian motion B with $B_0 = 0$, the process X_t defined $X_t = F_x(B)_t$ is the unique solution of $E_x(\sigma, b)$; furthermore, if U is an $\mathcal{F}_0$-measurable real-valued random variable, the process $F_U(B)_t$ is the unique solution with $X_0 = U$.

Remark)

Assertion (iii) implies in particular that weak uniqueness holds for $E(\sigma, b)$: any solution of $E_x(\sigma, b)$ must be of the form $F_x(B)$ and its law is thus uniquely determined as the image of $W(dw)$ under F_x.

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Proof of Theorem 8.5

Again we consider only the case $d = m = 1$. Let $\mathcal{N}$ be the class of all W-negligible sets in $C(\mathbb{R}_+, \mathbb{R})$, and, for every $t \in [0, \infty]$, define

$$\mathcal{G}_t = \sigma(w(s) : s \in [0, t]) \vee \mathcal{N}.$$

For every $x \in \mathbb{R}$, we write X^x for the solution of $E_x(\sigma, b)$ corresponding to the filtered probability space $(C(\mathbb{R}_+, \mathbb{R}), \mathcal{G}_\infty, (\mathcal{G}_t), W)$ and the (canonical) Brownian motion $B_t(w) = w(t)$. This solution exists and is unique (up to indistinguishability) by Theorem 8.3, noting that the filtration $(\mathcal{G}_t)$ is complete by construction.

Let $x, y \in \mathbb{R}$ and let T_n be the stopping time defined by

$$T_n = \inf\{t \geq 0 : |X_t^x| \vee |X_t^y| \geq n\}.$$

Let $p \geq 2$ and $T \geq 1$. Using the Burkholder-Davis-Gundy inequalities, and then the Hölder inequality, we get, for $t \in [0, T]$,

Proof of Theorem 8.5 (cont)

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{s\le t}|X^x_{s\wedge T_n}-X^y_{s\wedge T_n}|^p\right]\\
&\le C_p\Bigg(|x-y|^p+\mathbb{E}\left[\sup_{s\le t}\Big|\int_0^{s\wedge T_n}(\sigma(r,X^x_r)-\sigma(r,X^y_r))dB_r\Big|^p\right]\\
&\quad + E\left[\sup_{s\le t}\Big|\int_0^{s\wedge T_n}(b(r,X^x_r)-b(r,X^y_r))dr\Big|^p\right]\Bigg)\\
&\le C_p\Bigg(|x-y|^p+C'_p\mathbb{E}\left[\left(\int_0^{t\wedge T_n}(\sigma(r,X^x_r)-\sigma(r,X^y_r))^2dr\right)^{p/2}\right]\\
&\quad + E\left[\left(\int_0^{t\wedge T_n}|b(r,X^x_r)-b(r,X^y_r)|dr\right)^p\right]\Bigg)
\end{aligned}
$$

Proof of Theorem 8.5 (cont)

$$\le C_p \left(|x-y|^p + C'_p t^{\frac{p}{2}-1} \mathbb{E}\left[\int_0^t |\sigma(r\wedge T_n, X^x_{r\wedge T_n}) - \sigma(r\wedge T_n, X^y_{r\wedge T_n})|^p dr \right] \right.$$

$$\left. + t^{p-1} \mathbb{E}\left[\int_0^t |b(r\wedge T_n, X^x_{r\wedge T_n}) - b(r\wedge T_n, X^y_{r\wedge T_n})|^p dr \right] \right)$$

$$\le C''_p \left(|x-y|^p + T^p \int_0^t \mathbb{E}[|X^x_{r\wedge T_n} - X^y_{r\wedge T_n}|^p dr] \right),$$

where the constants C_p, C'_p, C''_p depend on p (and on the constant K appearing in our assumption on σ and b) but not on n or on x; y and T.

Proof of Theorem 8.5 (cont)

As the function $t \mapsto \mathbb{E}[\sup_{s\leq t} |X^x_{s\wedge T_n} - X^y_{s\wedge T_n}|^p]$ is bounded, Grownall's lemma implies that, for $t \in [0, T]$,

$$\mathbb{E}\left[\sup_{s\leq t} |X^x_{s\wedge T_n} - X^y_{s\wedge T_n}|^p\right] \leq C''_p |x-y|^p \exp(C''_p T^p t),$$

hence, letting n tend to ∞,

$$\mathbb{E}\left[\sup_{s\leq t} |X^x_s - X^y_s|^p\right] \leq C''_p |x-y|^p \exp(C''_p T^p t).$$

The topology on the space $C(\mathbb{R}_+, \mathbb{R})$ is defined by the distance

$$\mathbf{d}(w, w') = \sum_{k=1}^{\infty} \alpha_k \left(\sup_{s\leq k} |w(s) - w'(s)| \wedge 1 \right)$$

where the sequence of positive reals α_k can be chosen arbitrarily, provided that $\sum_{k=1}^{\infty} \alpha_k$ converges.

Proof of Theorem 8.5 (cont)

We may choose α_k so that

$$\sum_{k=1}^{\infty} \alpha_k \exp(C''_p k^{p+1}) < \infty.$$

For every $x \in \mathbb{R}$, we consider X^x as a random variable with values in $C(\mathbb{R}_+, \mathbb{R})$. The preceding estimates and Jensen's inequality then show that

$$\mathbb{E}[(\mathbf{d}(X^x, X^y))^p] \leq \left(\sum_{k=1}^{\infty} \alpha_k\right)^{p-1} \sum_{k=1}^{\infty} \alpha_k \mathbb{E}\left[\sup_{s \leq k} |X_s^x - X_s^y|^p\right] \leq \bar{C}_p |x-y|^p,$$

with a constant $\bar{C}_p$ independent of x and y. By Kolmogorov's continuity theorem applied to the process $(X^x : x \in \mathbb{R})$ with values in the space $E = C(\mathbb{R}_+, \mathbb{R})$ equipped with the distance $\mathbf{d}$, we get that $(X^x : x \in \mathbb{R})$ has a modification with continuous sample paths, which we denote by $(\widetilde{X}^x : x \in \mathbb{R})$. We set $F_x(w) = \widetilde{X}^x = (\widetilde{X}^x_t)_{t \geq 0}$, Property (ii) is then obvious.

Proof of Theorem 8.5 (cont)

The mapping $w \mapsto F_x(w)$ is measurable from $C(\mathbb{R}_+, \mathbb{R})$ equipped with the σ-field $\mathcal{G}_\infty$ into $C(\mathbb{R}_+, \mathbb{R})$ equipped with the Borel σ-field $\mathcal{C} = \sigma(w(s) : s \geq 0)$. Moreover, for every $t \geq 0$, $F_x(w)_t = \widetilde{X}_t^x(w) = X_t^x(w)$ a.s. is $\mathcal{G}_t$-measurable hence coincides $W(dw)$ a.s. with a measurable function of $(w(s) : 0 \leq s \leq t)$. Thus property (i) holds.