# Math 562 Fall 2020

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# Outline

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HW6 is due Friday, 11/20, at noon. Please submit your homework via the course Moodle page. Make sure that your HW is uploaded successfully.

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In this chapter, we study stochastic differential equations. The goal of stochastic differential equations is to provide a model for a differential equation perturbed by a random noise. Consider an ordinary differential equation of the form

$$dy_t = b(y_t)dt.$$

Such an equation is used to model the evolution of a physical system.

If we take random perturbations of the system into account, we add a noise term, which is typically of the form $\sigma dB_t$, where $B$ denotes a Brownian motion, and $\sigma$ is a constant corresponding to the intensity of the noise. Note that the use of Brownian motion here is justified by its property of independence of increments, corresponding to the fact that the random perturbations affecting disjoint time intervals are assumed to be independent.

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In this way, we arrive at a stochastic differential equation of the form

$$dy_t = b(y_t)dt + \sigma dB_t,$$

or in integral form,

$$y_t = y_0 + \int_0^t b(y_s)ds + \sigma B_t.$$

We generalize the preceding equation by allowing $\sigma$ to depend on the state of the system at time $t$:

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We can still generalize the preceding equation by allowing $\sigma$ and $b$ to depend on the time parameter $t$. This leads to the following definition.

**Definition 8.1**

Let $d$ and $m$ be positive integers, and let $\sigma$ and $b$ be bounded Borel functions defined on $\mathbb{R}_+ \times \mathbb{R}^d$ and taking values in $M_{d\times m}(\mathbb{R})$ and $\mathbb{R}^d$ respectively, where $M_{d\times m}(\mathbb{R})$ is the set of all $d \times m$ matrices with real entries. We write $\sigma = (\sigma_{ij})_{1\le i\le d, 1\le j\le m}$ and $b = (b_i)_{1\le i\le d}$.

A solution of the stochastic differential equation

$$dX_t = \sigma(t, X_t)dB_t + b(t, X_t)dt, \qquad E(\sigma, b)$$

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consists of:

## Definition 8.1 (cont)

- a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty]}, \mathbb{P})$ (where the filtration is always assumed to be complete);
- an $m$-dimensional $(\mathcal{F}_t)$-Brownian motion $B = (B^1, \ldots, B^m)$ started from 0;
- an $(\mathcal{F}_t)$-adapted process $X = (X^1, \ldots, X^d)$ taking values in $\mathbb{R}^d$, with continuous sample paths, such that

$$X_t = X_0 + \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds$$

meaning that, for every $i \in \{1, \ldots, d\}$,

$$X_t^i = X_0^i + \sum_{j=1}^m \int_0^t \sigma_{ij}(s, X_s) dB_s^j + \int_0^t b_i(s, X_s) ds.$$

If additionally $X_0 = x \in \mathbb{R}^d$, we say that $X$ is a solution of $E_x(\sigma, b)$.

Note that, when we speak about a solution of $E(\sigma, b)$, we do not fix a priori the filtered probability space and the Brownian motion B. When we fix these objects, we will say so explicitly.

There are several notions of existence and uniqueness for stochastic differential equations.

**Definition 8.2**

For the equation $E(\sigma, b)$, we say that there is

- weak existence if, for every $x \in \mathbb{R}^d$, there exists a solution of $E_x(\sigma, b)$;
- weak existence and weak uniqueness if in addition, for every $x \in \mathbb{R}^d$, all solutions of $E_x(\sigma, b)$ have the same law;
- pathwise uniqueness if, whenever the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty]}, \mathbb{P})$ and the $(\mathcal{F}_t)$-Brownian motion $B$ are fixed, two solution $X$ and $X'$ such that $X_0 = X'_0$ a.s. are indistinguishable.

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## Definition 8.2 (cont)

Furthermore, we say that a solution $X$ of $E_x(\sigma, b)$ is a strong solution if $X$ is adapted with respect to the completed canonical filtration of $B$.

## Remark

It may happen that weak existence and weak uniqueness hold but pathwise uniqueness fails. For a simple example, consider a 1-dim Brownian motion $\beta$ started from $\beta_0 = y$, and set

$$B_t = \int_0^t \operatorname{sgn}(\beta_s) d\beta_s,$$

where $\operatorname{sgn}(x) = 1$ if $x > 0$ and $\operatorname{sgn}(x) = -1$ if $x \leq 0$. Then, one immediately gets from the “associativity” of stochastic integrals that

$$\beta_t = y + \int_0^t \operatorname{sgn}(\beta_s) dB_s.$$

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Moreover, $B$ is a continuous martingale with quadratic variation $\langle B, B \rangle_t = t$, and Theorem 5.12 shows that $B$ is a Brownian motion started from 0. We thus see that $\beta$ solves the stochastic differential equation

$$dX_t = \mathrm{sgn}(X_t) dB_t, \qquad X_0 = y,$$

and it follows that weak existence holds for this equation. Theorem 5.12 again shows that any other solution of this equation must be a Brownian motion, which gives weak uniqueness. On the other hand, pathwise uniqueness fails. In fact, taking $y = 0$ in the construction, one easily sees that both $\beta$ and $-\beta$ solve the preceding stochastic differential equation with the same Brownian motion $B$ and initial value 0 (note that $\int_0^t 1_{\{\beta_s = 0\}} ds = 0$, which implies $\int_0^t 1_{\{\beta_s = 0\}} dB_s = 0$). One can also show that $\beta$ is not a strong solution: One verifies that the canonical filtration of $B$ coincides with the canonical filtration of $|\beta|$, which is strictly smaller than that of $\beta$.

The next theorem links the different notions of existence and uniqueness.

**Theorem (Yamada-Watanabe)**

If both weak existence and pathwise uniqueness hold, then weak uniqueness also holds. Moreover, for any choice of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty]}, \mathbb{P})$ and of the $(\mathcal{F}_t)$-Brownian motion $B$, there exists, for every $x \in \mathbb{R}^d$, a (unique) strong solution of $E_x(\sigma, b)$.

The proof of this theorem is pretty complicated. See the book by Karatzas and Shreve for a proof.

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In this section, we work under the following assumptions.

**Assumptions**

The functions $\sigma$ and $b$ are continuous on $\mathbb{R}_+ \times \mathbb{R}^d$ and Lipschitz in the variable $x$: There exists a constant $K$ such that, for every $t \geq 0, x, y \in \mathbb{R}^d$,

$$|\sigma(t,x) - \sigma(t,y)| \leq K|x-y|;$$
$$|b(t,x) - b(t,y)| \leq K|x-y|.$$

**Theorem 8.3**

Under the preceding assumptions, pathwise uniqueness holds for $E(\sigma, b)$, and, for every choice of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty]}, \mathbb{P})$ and of the $(\mathcal{F}_t)$-Brownian motion $B$, for every $x \in \mathbb{R}^d$, there exists a (unique) strong solution of $E_x(\sigma, b)$.

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The theorem implies in particular that weak existence holds for $E(\sigma, b)$. Weak uniqueness will follow from the next theorem (it can also be deduced from pathwise uniqueness using the Yamada-Watanabe theorem).

**Remark**

One can "localize" the Lipschitz assumption on $\sigma$ and $b$, meaning that the constant $K$ may depend on the compact set on which the parameters $t$ and $x; y$ are considered. In that case, it is, however, necessary to keep a condition of linear growth of the form

$$|\sigma(t,x)| \leq K(1+|x|), \quad |b(t,x)| \leq K(1+|x|).$$

This kind of condition, which avoids the blow-up of solutions, already appears in ordinary differential equations.

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## Proof of Proposition 8.3

We we consider only the case $d = m = 1$. The general case follows the same argument, only the notations are more complicated. Let us start by proving pathwise uniqueness. We consider (on the same filtered probability space, with the same Brownian motion $B$) two solutions $X$ and $X'$ such that $X_0 = X'_0$. Fix $M > 0$ and set

$$\tau = \inf\{t \geq 0 : |X_t| \geq M \text{ or } |X'_t| \geq M\}.$$

Then, for every $t \geq 0$,

$$X_{t\wedge\tau} = X_0 + \int_0^{t\wedge\tau} \sigma(s, X_s)dB_s + \int_0^{t\wedge\tau} b(s, X_s)ds$$

and an analogous equation holds for $X'_{t\wedge\tau}$. Fix a constant $T > 0$. By considering the difference between the two equations, we get, for $t \in [0, T]$,

## Proof of Proposition 8.3 (cont)

$$
\begin{aligned}
&\mathbb{E}[(X_{t\wedge\tau} - X'_{t\wedge\tau})^2] \\
&\leq 2\mathbb{E}\left[\left(\int_0^{t\wedge\tau}(\sigma(s,X_s) - \sigma(s,X'_s))dB_s\right)^2\right] \\
&\quad + 2\mathbb{E}\left[\left(\int_0^{t\wedge\tau}(b(s,X_s) - b(s,X'_s))ds\right)^2\right] \\
&\leq 2\mathbb{E}\left[\int_0^{t\wedge\tau}(\sigma(s,X_s) - \sigma(s,X'_s))^2 ds\right] \\
&\quad + 2T\mathbb{E}\left[\int_0^{t\wedge\tau}(b(s,X_s) - b(s,X'_s))^2 ds\right] \\
&\leq 2K^2(1+T)\mathbb{E}\left[\int_0^{t\wedge\tau}(X_s - X'_s)^2 ds\right] \leq 2K^2(1+T)\mathbb{E}\left[\int_0^{t}(X_{s\wedge\tau} - X'_{s\wedge\tau})^2 ds\right]
\end{aligned}
$$

## Proof of Proposition 8.3 (cont)

Hence the function $h(t) = \mathbb{E}[(X_{t\wedge\tau} - X'_{t\wedge\tau})^2]$ satisfies

$$h(t) \le C \int_0^t h(s)ds, \quad t \in [0, T],$$

where $C = 2K^2(1 + T)$. Now we are going to use

## Lemma 8.4 (Gronwall's lemma)

Let $T > 0$ and let $g$ be a non-negative bounded measurable function on $[0, T]$. Assume that there exist two constants $a \ge 0$ and $b \ge 0$ such that, for every $t \in [0, T]$,

$$g(t) \le a + b \int_0^t g(s)ds,$$

Then, we also have, for every $t \in [0, T]$,

$$g(t) \le ae^{bt}.$$

## Proof of Proposition 8.3 (cont)

Hence the function $h(t) = \mathbb{E}[(X_{t\wedge\tau} - X'_{t\wedge\tau})^2]$ satisfies

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### Proof of Theorem 8.3 (cont)

The function h is bounded above by $4M^2$ and the assumption of the lemma holds with $a = 0$ and $b = C$. We thus get $h = 0$, so that $X_{t\wedge\tau} = X'_{t\wedge\tau}$. By letting $M$ tend to $\infty$, we get $X_t = X'_t$, which completes the proof of pathwise uniqueness.

For the second assertion, we construct a solution using Picard's approximation method. We will deal with this next time.