Math 562 Fall 2020

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Outline

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HW6 is due Friday, 11/20, at noon. Please submit your homework via the course Moodle page. Make sure that your HW is uploaded successfully.

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Consider again a bounded domain D and a continuous function g on ∂D. Let $T = \inf\{t \geq 0 : B_t \notin D\}$ be the exit time of D by Brownian motion. Proposition 7.7 (i) shows that the solution of the Dirichlet problem in D with boundary condition g, if it exists, is given by

$$u(x) = \mathbb{E}_x[g(B_T)] = \int_{\partial D} \omega(x, dy) g(y),$$

where, for every $x \in D$, $\omega(x, dy)$ denotes the distribution of B_T under $\mathbb{P}_x$. The measure $\omega(x, dy)$ is a probability measure on ∂D called the harmonic measure of D relative to x. In general, it is hopeless to try to find an explicit expression for the measures $\omega(x, dy)$. It turns out that, in the case of balls, such an explicit expression is available and makes the representation of solutions of the Dirichlet problem more concrete.

From now on, we suppose that $D = B(0, 1)$ is the open unit ball in $\mathbb{R}^d$. We also assume that $d \geq 2$ to avoid trivialities. The boundary $\partial B(0, 1)$ is the unit sphere.

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Definition 7.10

The Poisson kernel (of the unit ball) is the function K defined on $B(0,1) \times \partial B(0,1)$ by

$$K(x,y) = \frac{1-|x|^2}{|y-x|^d}, \quad x \in B(0,1), y \in \partial B(0,1).$$

Lemma 7.11

For every fixed $y \in \partial B(0,1)$, the function $x \mapsto K(x,y)$ is harmonic on $B(0,1)$.

Proof

Define $K_y(x) = K(x,y), x \in B(0,1)$. $K_y(\cdot)$ is a C^∞ function on $B(0,1)$. It is elementary to check that $K_y(\cdot)$ is harmonic on $B(0,1)$.

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Lemma 7.12

Let $0 \leq r_1 < r_2$ and let $h : (r_1, r_2) \to \mathbb{R}$ be a measurable function. The function $u(x) = h(|x|)$ is harmonic on $\{x \in \mathbb{R}^d : r_1 < |x| < r_2\}$ if and only if there exist two constants a and b such that

$$h(r) = \begin{cases} a + b\log r, & d = 2, \\ a + br^{2-d}, & d \geq 3. \end{cases}$$

Proof of Lemma 7.12

Suppose that $u(x) = h(|x|)$ is harmonic on $\{x \in \mathbb{R}^d : r_1 < |x| < r_2\}$, Then u is twice continuously differentiable and so is h. From the expression of the Laplacian of a radial function, we get that $\Delta u = 0$ if and only if

$$h''(r) + \frac{d-1}{r} h'(r) = 0, \quad r \in (r_1, r_2).$$

The solutions of this second order linear differential equations are the functions of the form given in the statement. The lemma follows.

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Recall that $\sigma_1(dy)$ is the normalized surface measure on the unit sphere.

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For every $x \in B(0,1)$,

$$\int_{\partial B(0,1)} K(x,y)\sigma_1(dy) = 1.$$

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For every $x \in B(0,1)$, define

$$F(x) = \int_{\partial B(0,1)} K(x,y)\sigma_1(dy).$$

Then the preceding lemma implies that F is harmonic on $B(0,1)$.

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Proof of Lemma 7.13 (cont)

Indeed, if $x \in B(0,1)$ and $r < 1 - |x|$, Lemma 7.11 and the mean value property imply that, for every $y \in \partial B(0,1)$,

$$K(x,y) = \int K(z,y)\sigma_{x,r}(dz).$$

By Fubini,

$$\int F(z)\sigma_{x,r}(dz) = \int \left(\int K(z,y)\sigma_1(dy) \right) \sigma_{x,r}(dz)$$
$$= \int \left(\int K(z,y)\sigma_{x,r}(dz) \right) \sigma_1(dy) = \int K(x,y)\sigma_1(dy) = F(x)$$

showing that the mean value property holds for F.

Proof of Lemma 7.13 (cont)

If ψ is a vector isometry of $\mathbb{R}^d$, we have $K(\psi(x), \psi(y)) = K(x, y)$ for every $x \in B(0,1)$ and $y \in \partial B(0,1)$, and the fact that $\sigma_1(dy)$ is invariant under ψ implies that $F(\psi(x)) = F(x)$ for every $x \in B(0,1)$. Hence F is a radial harmonic function and Lemma 7.12 (together with the fact that F is bounded in the neighborhood of 0) implies that F is constant. Since $F(0) = 1$, the proof is complete.

Theorem 7.14

Let g be a continuous function on $\partial B(0,1)$. The unique solution of the Dirichlet problem in $B(0,1)$ with boundary condition g is given by

$$u(x) = \int_{\partial B(0,1)} K(x,y)\sigma_1(dy), \quad x \in B(0,1).$$

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Proof of Theorem 7.14

Repeating the argument in the first paragraph of the proof of Lemma 7.13, we get that u is harmonic on $B(0,1)$. To verify the boundary condition, fix $y_0 \in \partial B(0,1)$. For every $\delta > 0$, the explicit form of the Poisson kernel shows that, if $x \in B(0,1)$ and $y \in \partial B(0,1)$ with $|x - y_0| < \delta/2$ and $|y - y_0| > \delta$, then

$$K(x,y) \leq \left(\frac{2}{\delta}\right)^d (1 - |x|^2).$$

It follows from this bound that, for every $\delta > 0$,

$$\lim_{B(0,1)\ni x \to y_0} \int_{\{|y-y_0|>\delta\}} K(x,y)\sigma_1(dy) = 0. \tag{1}$$

Then, given $\epsilon > 0$, we can choose $\delta > 0$ sufficiently small so that the conditions $y \in \partial B(0,1)$ and $|y - y_0| \leq \delta$ imply $|g(y) - g(y_0)| \leq \epsilon$.

Proof of Theorem 7.14 (cont)

If $M = \|g\|_\infty$, it follows that

$$
\begin{aligned}
|u(x) - g(y_0)| &= \Big| \int K(x,y)(g(y) - g(y_0)\sigma_1(dy) \Big| \\
&\leq 2M \int_{\{|y-y_0|>\delta\}} K(x,y)\sigma_1(dy) + \epsilon
\end{aligned}
$$

using Lemma 7.13 in the first equality, and then our choice of δ. Thanks to (1), we now get

$$\lim_{x\to y_0} \sup_{x\in B(0,1)} |u(x) - g(y_0)| \leq \epsilon.$$

Since ϵ was arbitrary, this yields the desired boundary condition.

The preceding theorem allows us to identify the harmonic measures of the unit ball.

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Corollary 7.15

Let $T = \inf\{t \geq 0 : B_t \notin B(0,1)\}$. For every $x \in B(0,1)$, the distribution of B_T under $\mathbb{P}_x$ has density $K(x,y)$ with respect to $\sigma_1(dy)$.

This is immediate since, by combining Proposition 7.7 (i) with Theorem 7.14, we get that, for any continuous function g on $\partial B(0,1)$,

$$\mathbb{E}_x[g(B_T)] = \int K(x,y) g(y) \sigma_1(dy), \quad x \in B(0,1).$$

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We consider again a d-dimensional Brownian motion $(B_t)_{t\geq 0}$ that starts from x under the probability measure $\mathbb{P}_x$. We again suppose that $d \geq 2$, since the corresponding results for $d = 1$ have already been derived in the previous chapters.

For every $a \geq 0$, we introduce the stopping time

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Proposition 7.16

Suppose that $x \neq 0$, and let ϵ and R be such that $0 < \epsilon < |x| < R$. Then

$$\mathbb{P}_x(U_\epsilon < U_R) = \begin{cases} \frac{\log R - \log|x|}{\log R - \log \epsilon}, & d = 2, \\ \frac{R^{2-d} - |x|^{2-d}}{R^{2-d} - \epsilon^{2-d}}, & d \geq 3. \end{cases} \quad (2)$$

Consequently, we have $\mathbb{P}_x(U_0 < \infty) = 0$ and for every $\epsilon \in (0, |x|)$,

$$\mathbb{P}_x(U_\epsilon < \infty) = \begin{cases} 1, & d = 2, \\ \left(\frac{\epsilon}{|x|}\right)^{d-2}, & d \geq 3. \end{cases}$$

Proof of Proposition 7.16

Define $D_{\epsilon,R} = \{y \in \mathbb{R}^d : \epsilon < |y| < R\}$. Let $u(x)$ be the function defined for $x \in D_{\epsilon,R}$ by the rhs of (2). By Lemma 7.12, u is harmonic on $D_{\epsilon,R}$, and it is also clear that u solves the Dirichlet problem in $D_{\epsilon,R}$ with boundary condition $g(y) = 0$ if $|y| = R$ and $g(y) = 1$ if $|y| = \epsilon$.

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Proof of Proposition 7.16 (cont)

If $T_{\epsilon,R}$ denotes the first exit time from $D_{\epsilon,R}$, Proposition 7.7 shows that we must have $u(x) = \mathbb{E}_x[g(B_{T_{\epsilon,R}})]$ for $x \in D_{\epsilon,R}$. Formula (2) follows since $\mathbb{E}_x[g(B_{T_{\epsilon,R}})] = \mathbb{P}_x(U_\epsilon < U_R)$.

If $R > |x|$ is fixed, the event $\{U_0 < U_R\}$ is ($\mathbb{P}_x$-a.s) contained in $\{U_\epsilon < U_R\}$, for every $\epsilon \in (0, |x|)$. By passing to the limit $\epsilon \to 0$ in the right-hand side of (2), we thus get that $\mathbb{P}_x(U_0 < U_R) = 0$. Since $U_R \uparrow \infty$ as $R \uparrow \infty$, it follows that $\mathbb{P}_x(U_0 < \infty) = 0$.

Finally, we have also $\mathbb{P}_x(U_\epsilon < \infty) = \lim_{R\to\infty} \mathbb{P}_x(U_\epsilon < U_R)$, and by letting $R \to \infty$ in the right-hand side of (2) we get the stated formula for $\mathbb{P}_x(U_\epsilon < \infty)$.

For every $y \in \mathbb{R}^d$, define $\tau_y = \inf\{t \geq 0 : B_t = y\}$, so that in particular $\tau_0 = U_0$. The property $\mathbb{P}_x(\tau_0 < \infty) = 0$ for $x \neq 0$ implies that $\mathbb{P}_x(\tau_y < \infty) = 0$ whenever $x \neq y$, by translation invariance. This means that the probability for Brownian motion to visit a fixed point other than its starting point is zero: one says that points are polar for d-dimensional Brownian motion with $d \geq 2$.

Let m denotes Lebesgue measure on $\mathbb{R}^d$, it follows from Fubini's theorem that

$$\mathbb{E}_x[m(\{B_t : t \geq 0\})] = \mathbb{E}_x\left[\int_{\mathbb{R}^d} dy 1_{\{\tau_y < \infty\}}\right] = \int_{\mathbb{R}^d} dy \mathbb{P}_x(\tau_0 < \infty) = 0,$$

and therefore $m(\{B_t : t \geq 0\}) = 0$ $\mathbb{P}_x$-a.s. One can nonetheless prove that the Hausdorff dimension of the curve $\{B_t : t \geq 0\}$ is equal to 2 in any dimension $d \geq 2$. In some sense, this shows that the planar Brownian curve is “not so far” from having positive Lebesgue measure.

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Theorem 7.17

(i) In dimension $d = 2$, Brownian motion is recurrent, meaning that almost surely, for every nonempty open subset O of $\mathbb{R}^d$, the set $\{t \geq 0 : B_t \in O\}$ is unbounded.

(ii) In dimension $d \geq 3$, Brownian motion is transient, meaning that

$$\lim_{t\to\infty} |B_t| = \infty, \quad a.s.$$

Proof of Theorem 7.17

(i) Suffices to prove that the statement holds when O is an open ball of rational radius centered at a point with rational coordinates. So it suffices to consider a fixed open ball $\mathcal{B}$ and we may assume that $\mathcal{B}$ is centered at 0 and that the starting point of B is $x \neq 0$. By Proposition 7.16 we know that Brownian motion will never hit 0, but still will hit any open ball centered at 0. It follows that B must visit $\mathcal{B}$ at arbitrarily large times, a.s.

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Proof of Theorem 7.17 (cont)

(ii) Again we can assume that the starting point of B is $x \neq 0$. Since the function $x \to |x|^{2-d}$ is harmonic on $\mathbb{R}^d \setminus \{0\}$, and since we saw that B does not hit 0, we get that $|B_t|^{2-d}$ is a local martingale and hence a supermartingale by Proposition 4.7. By Theorem 3.19 (and the fact that a positive supermartingale is automatically bounded in L^1), we know that $|B_t|^{2-d}$ converges a.s. as $t \to \infty$. The a.s. limit must be zero (otherwise the curve $\{B_t : t \geq 0\}$ would be bounded!) and this says exactly that B_t converges to ∞ as $t \to \infty$.

Remark

In dimension $d = 2$, one can (slightly) reinforce the recurrence property by saying that a.s. for every nonempty open subset O of $\mathbb{R}^2$, the Lebesgue measure of $\{t \geq 0 : B_t \in O\}$ is infinite. This is a consequence of the strong Markov property.

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