Math 562 Fall 2020

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Outline

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HW6 is due Friday 11/20 at noon. Please submit your homework via the course Moodle page. Make sure that your HW is uploaded successfully.

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In the remainder of this section, we assume that the domain D is bounded.

Definition 7.6 (Classical Dirichlet problem)

Let g be a continuous function on ∂D. A function $u : D \to \mathbb{R}$ solves the Dirichlet problem in D with boundary condition g, if u is harmonic on D and has boundary condition g, in the sense that, for every $y \in \partial D$, $\lim_{D \ni x \to y} u(x) = g(y)$.

If u solves the Dirichlet problem with boundary condition g, the function $\tilde{u}$ defined on $\overline{D}$ by $\tilde{u}(x) = u(x)$ if $x \in D$, and $\tilde{u}(x) = g(x)$ if $x \in \partial D$, is then continuous, hence bounded on $\overline{D}$.

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Proposition 7.7

Let D be a bounded domain, and write $T = \inf\{t \geq 0 : B_t \notin D\}$ for the exit time of Brownian motion from D.

(i) Let g be a continuous function on ∂D, and let u be a solution of the Dirichlet problem in D with boundary condition g. Then, for every $x \in D$,

$$u(x) = \mathbb{E}_x[g(B_T)].$$

(ii) Let g be a bounded measurable function on ∂D. Then the function

$$u(x) = \mathbb{E}_x[g(B_T)], \quad x \in D$$

is harmonic on D.

Remark

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Proof of Proposition 7.7

(i) Fix $x \in D$ and $\epsilon_0 > 0$ such that $\overline{B(x, \epsilon_0)} \subset D$. For every $\epsilon \in (0, \epsilon_0)$, let D_ϵ be the connected component containing x of the open set consisting of all points of D whose distance to D^c is greater than ϵ. If $T_\epsilon = \inf\{t \geq 0 : B_t \notin D_\epsilon\}$, Proposition 7.3 shows that

$$u(x) = \mathbb{E}_x[u(B_{T_\epsilon})].$$

Now observe that $T_\epsilon \uparrow T$ as $\epsilon \downarrow 0$ (if T' is the increasing limit of T_ϵ as $\epsilon \downarrow 0$, we have $T' \leq T$ and on the other hand $B_{T'} \in \partial D$ by the continuity of sample paths). Using dominated convergence, it follows that $\mathbb{E}_x[u(B_{T_\epsilon})]$ converges to $\mathbb{E}_x[u(B_T)]$ as $\epsilon \downarrow 0$.

Proof of Proposition 7.7 (cont)

(ii) By Lemma 7.5, it suffices to show that the function $u(x) = \mathbb{E}_x[g(B_T)]$ satisfies the mean value property. Recall the $T_{x,r} = \inf\{t \geq 0 : |B_t - x| = r\}$ for $x \in \mathbb{R}^d$ and $r > 0$. Fix $x \in D$ and $r > 0$ so that $\overline{B(x,r)} \subset D$. We now apply the strong Markov property: let $\Phi(w)$, for $w \in C(\mathbb{R}_+, \mathbb{R}^d)$ with $w(0) \in D$, be the value of g at the first exit point of w from D (we let $\Phi(w) = 0$ if w never exits D) and we observe that we have

$$g(B_T) = \Phi(B_t, t \geq 0) = \Phi(B_{T_{x,r}+t}, t \geq 0)$$

because the paths $(B_t, t \geq 0)$ and $(B_{T_{x,r}+t}, t \geq 0)$ have the same exit point from D. It follows that

$$\begin{aligned} u(x) = \mathbb{E}_x[g(B_T)] &= \mathbb{E}_x[\Phi(B_{T_{x,r}+t}, t \geq 0)] \\ &= \mathbb{E}_x[\mathbb{E}_{B_{T_{x,r}}}[\Phi(B_t, t \geq 0)]] = \mathbb{E}_x[u(B_{T_{x,r}})] \end{aligned}$$

Since the law of $B_{T_{x,r}}$ under $\mathbb{P}_x$ is $\sigma_{x,r}$, this gives the mean value property.

Part (i) of Proposition 7.7 tells us that the solution of the Dirichlet problem with boundary condition g, if it exists, is given by the probabilistic formula $u(x) = \mathbb{E}_x[g(B_T)]$. On the other hand, for any choice of the (bounded measurable) function g on ∂D, part (ii) tells us that the probabilistic formula yields a function u that is harmonic on D. Even if g is assumed to be continuous, it is however not clear that the function u has boundary condition g, and this need not be true in general. We now give a theorem that partially answers his question.

If $y \in \partial D$, we say that D satisfies the exterior cone condition at y if there exist a (nonempty) open cone $\mathcal{C}$ with vertex at y and $r > 0$ such that the intersection of $\mathcal{C}$ with $B(y, r)$ is contained in D^c. For instance, a convex domain satisfies the exterior cone condition at every point of its boundary.

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Theorem 7.8 (Solution of the Dirichlet problem)

Let D be a bounded domain in $\mathbb{R}^d$. Assume that D satisfies the exterior cone condition at every $y \in \partial D$. Then, for every continuous function g on ∂D, the fomula

$$u(x) = \mathbb{E}_x[g(B_T)], \quad x \in D,$$

where $T = \inf\{t \geq 0 : B_t \notin D\}$, gives the unique solution of the Dirichlet problem with boundary condition g.

Proof of Theorem 7.8

Thanks to Proposition 7.7 (ii), we only need to verify that, for every fixed $y \in \partial D$,

$$\lim_{D \ni x \to y} u(x) = g(y). \tag{1}$$

Let $\epsilon > 0$. Since g is continuous, we can find $\delta > 0$ such that $|g(z) - g(y)| \leq \epsilon/3$ whenever $z \in \partial D$ and $|z - y| < \delta$.

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Proof of Theorem 7.8 (cont)

Let $M > 0$ be such that $|g(z)| \leq M$ for every $z \in \partial D$. Then for every $\eta > 0$,

$$\begin{aligned}
&|u(x) - g(y)| \\
\leq &\mathbb{E}_x[|g(B_T) - g(y)|1_{\{T\leq\eta\}}] + \mathbb{E}_x[|g(B_T) - g(y)|1_{\{T>\eta\}}] \\
\leq &\mathbb{E}_x[|g(B_T) - g(y)|1_{\{T\leq\eta\}}1_{\{\sup\{|B_t-x|:t\leq\eta\}\leq\delta/2\}}] \\
&+ 2M\mathbb{P}_x(\sup_{t\leq\eta}|B_t - x| > \frac{\delta}{2}) + 2M\mathbb{P}_x(T > \eta) \\
=:&A_1 + A_2 + A_3.
\end{aligned}$$

We assume that $|y - x| < \delta/2$, and we bound these three terms separately.

First note that we have $|B_T - y| \leq |B_T - x| + |y - x| < \delta$ on the event

$$\{T \leq \eta\} \cap \{\sup\{|B_t - x| : t \leq \eta\} \leq \delta/2\}$$

and our choice of δ ensures that $A_1 \leq \epsilon/3$.

Proof of Theorem 7.8 (cont)

Then, translation invariance gives

$$A_2 = 2M\mathbb{P}_0\left(\sup_{t\leq\eta}|B_t| > \frac{\delta}{2}\right)$$

which tends to 0 when $\eta \downarrow 0$ by the continuity of sample paths. So we can fix $\eta > 0$ so that $A_2 \leq \epsilon/3$.

Finally, we claim that we can choose $\alpha \in (0, \delta/2]$ small enough so that we also have $A_3 = 2M\mathbb{P}_x(T > \eta) \leq \epsilon/3$ whenever $|x - y| < \alpha$. It follows that $|u(x) - g(y)| \leq \epsilon$ whenever $|x - y| < \alpha$, thus completing the proof of (1). it only remains to prove our claim, which is the goal of the next lemma.

Lemma 7.9

Under the exterior cone condition, we have for every $y \in \partial D$ and every $\eta > 0$,

$$\lim_{D \ni x \to y} \mathbb{P}_x(T > \eta) = 0.$$

Proof of Lemma 7.9

For every $u \in \mathbb{R}^d$ with $|u| = 1$ and every $\gamma \in (0,1)$, consider the cone

$$\mathcal{C}(u, \gamma) = \{z \in \mathbb{R}^d : z \cdot u > (1-\gamma)|z|\}.$$

If $y \in \partial D$ is given, the exterior cone condition means that we can fix $r > 0$, u and γ such that

$$y + (\mathcal{C}(u, \gamma) \cap B(0, r)) \subset D^c$$

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Proof of Lemma 7.9 (cont)

To simplify notation, we write $\mathcal{C} = \mathcal{C}(u, \gamma) \cap B(0, r)$ and

$$\mathcal{C}' = \mathcal{C}(u, \frac{\gamma}{2}) \cap B(0, \frac{r}{2}).$$

which is the intersection of a smaller cone with $B(0, \frac{r}{2})$.

For every open subset F of $\mathbb{R}^d$, define $T_F = \inf\{t \geq 0 : B_t \in F\}$. An application of Blumenthal's zero-one law along the lines of the proof of Proposition 2.14 (i) shows that $\mathbb{P}_0(T_{\mathcal{C}(u,\gamma/2)} = 0) = 1$ and hence $\mathbb{P}_0(T_{\mathcal{C}'} = 0) = 1$ by the continuity of sample paths.

$$A = \cap_{n=1}^{\infty} \{B_s \in \mathcal{C}(u, \gamma/2) \text{ for some } 0 < s \leq \frac{1}{n}\} \in \mathcal{F}_{0+}$$

$$\begin{aligned}\mathbb{P}_0(A) &= \lim_{n\to\infty} \mathbb{P}_0(B_s \in \mathcal{C}(u, \gamma/2) \text{ for some } 0 < s \leq \frac{1}{n}) \\ &\geq \mathbb{P}_0(B_{1/n} \in \mathcal{C}(u, \gamma/2)) > 0.\end{aligned}$$

Proof of Lemma 7.9 (cont)

On the other hand, define $\mathcal{C}'_a = \{z \in \mathcal{C}' : |z| > a\}$, for every $a \in (0, r/2)$. The sets $\mathcal{C}'_a$ increase to $\mathcal{C}'$ as $a \downarrow 0$, and thus we have $T_{\mathcal{C}'_a} \downarrow T_{\mathcal{C}'}$ as $a \downarrow 0$, $\mathbb{P}_0$-a.s. Hence, given any $\beta > 0$, we can fix a small enough so that

$$\mathbb{P}_0(T_{\mathcal{C}'_a} \leq \eta) \geq 1 - \beta.$$

Recalling that $y + \mathcal{C} \subset D^c$, we have

$$\mathbb{P}_x(T \leq \eta) \geq \mathbb{P}_x(T_{y+\mathcal{C}} \leq \eta) = \mathbb{P}_0(T_{y-x+\mathcal{C}} \leq \eta).$$

However, a simple geometric argument shows that, as soon as $|y - x|$ is small enough, the shifted cone $y - x + \mathcal{C}$ contains $\mathcal{C}'_a$ and therefore

$$\mathbb{P}_x(T \leq \eta) \geq \mathbb{P}_0(T_{\mathcal{C}'_a} \leq \eta) \geq 1 - \beta.$$

Since β was arbitrary, this completes the proof.