Math 562 Fall 2020

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Outline

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HW6 is due Friday 11/20 at noon. Please submit your homework via the course Moodle page. Make sure that your HW is uploaded successfully.

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Throughout this chapter, we let B stand for a d-dimensional Brownian motion that starts from x under the probability measure $\mathbb{P}_x$, for every $x \in \mathbb{R}^d$. Then $(B_t)_{t \geq 0}$ is a Feller process with semigroup

$$Q_t\varphi(x) = \int_{\mathbb{R}^d} (2\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{2t}\right) \varphi(y) dy,$$

for any $\varphi \in B(\mathbb{R}^d)$.

We write L for the generator of this Feller process. If ψ is a twice continuously differentiable function on $\mathbb{R}^d$ such that both ψ and $\Delta\psi$ belong to $C_0(\mathbb{R}^d)$, then $\psi \in D(L)$ and $L\psi = \frac{1}{2}\Delta\psi$.

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If $\varphi \in B(\mathbb{R}^d)$, then, for every fixed $t > 0$, $Q_t\varphi$ can be viewed as the convolution of φ with the C^∞ function

$$p_t(x) = \exp\left(-\frac{|x|^2}{2t}\right).$$

It follows that $Q_t\varphi$ is also a C^∞ function. Furthermore, if $\varphi \in C_0(\mathbb{R}^d)$, differentiation under the integral sign shows that all derivatives of $Q_t\varphi$ also belong to $C_0(\mathbb{R}^d)$. It follows that $Q_t\varphi \in D(L)$ and $L(Q_t\varphi) = \frac{1}{2}\Delta(Q_t\varphi)$.

Theorem 7.1

Let $\varphi \in C_0(\mathbb{R}^d)$. For every $t > 0$ and $x \in \mathbb{R}^d$, define

$$u_t(x) = Q_t\varphi(x) = \mathbb{E}_x[\varphi(B_t)].$$

Then, the function $u_t(x)$ solves the partial differential equation

$$\frac{\partial u_t}{\partial t} = \frac{1}{2}\Delta u_t,$$

on $(0, \infty) \times \mathbb{R}^d$. Furthermore, for every $x \in \mathbb{R}^d$,

$$\lim_{s \downarrow 0, y \to x} u_s(y) = \varphi(x).$$

Proof of Theorem 7.1

By the remarks preceding the theorem, we already know that, for every $t > 0$, u_t is a C^∞ function, $u_t \in D(L)$ and $Lu_t = \frac{1}{2}\Delta u_t$. Let $\epsilon > 0$. By applying Proposition 6.11 to $f = u_\epsilon$, we get for every $t \geq \epsilon$,

$$u_t = u_\epsilon + \int_0^{t-\epsilon} L(Q_s u_\epsilon)ds = u_\epsilon + \int_\epsilon^t Lu_s ds.$$

Since $Lu_s = Q_{s-\epsilon}(Lu_\epsilon)$ depends continuously on $s \in [\epsilon, \infty)$, it follows that, for $t \geq \epsilon$,

$$\frac{\partial u_t}{\partial t} = Lu_t = \frac{1}{2}\Delta u_t.$$

The last assertion is just the fact that $Q_s\varphi \to \varphi$ as $s \downarrow 0$.

Remark

We could have proved Theorem 7.1 by direct calculations from the explicit form of $Q_t\varphi$, and these calculations imply that the same statement holds if we only assume that φ is bounded and continuous. The above proof however has the advantage of showing the relation between this result and our general study of Markov processes. It also indicates that similar results will hold for more general equations of the form $\frac{\partial u}{\partial t} = Au$ provided A can be interpreted as the generator of an appropriate Markov process.

Brownian motion can be used to provide probabilistic representations for solutions of many other parabolic partial differential equations. In particular, solutions of equations of the form

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u - vu,$$

where v is a non-negative function on $\mathbb{R}^d$, are expressed via the so-called Feynman-Kac formula: See Exercise 7.28 below for a precise statement.

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Let us now turn to connections between Brownian motion and the Laplace equation $\Delta u = 0$. We start with a classical definition.

Definition 7.2

Let D be a domain of $\mathbb{R}^d$. A function $u : D \to \mathbb{R}$ is said to be harmonic on D if it is twice continuously differentiable and $\Delta u = 0$ on D.

Let D' be a subdomain of D with $\overline{D'} \subset D$. Consider the stopping time $T = \inf\{t \geq 0 : B_t \notin D'\}$. An application of Ito's formula (justified by the remark preceding Proposition 5.11) shows that, if u is harmonic on D, then for every $x \in D'$, the process

$$u(B_{t\wedge T}) = u(B_0) + \int_0^{t\wedge T} \nabla u(B_s) \cdot dB_s \qquad (1)$$

is a local martingale under $\mathbb{P}_x$.

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So, roughly speaking, harmonic functions are functions which when composed with Brownian motion give (local) martingales.

Proposition 7.3

Let u be harmonic on the domain D. Let D' be a bounded subdomain of D with $\overline{D'} \subset D$, and consider the stopping time $T = \inf\{t \geq 0 : B_t \notin D'\}$. Then, for every $x \in D'$,

$$u(x) = \mathbb{E}_x[u(B_T)].$$

Proof of Proposition 7.3

Since D' is bounded, both u and ∇u are bounded on D', and we also know that $\mathbb{P}_x(T < \infty) = 1$ for every $x \in D'$. It follows from (1) that $u(B_{t \wedge T})$ a (true) martingale, and in particular, we have

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Proof of Proposition 7.3 (cont)

By letting $t \uparrow \infty$ and using dominated convergence we get that

$$u(x) = \mathbb{E}_x[u(B_T)], \quad x \in D'.$$

The preceding proposition easily leads to the mean value property for harmonic functions. In order to state this property, first recall that the uniform probability measure on the unit sphere, denoted by $\sigma_1(dy)$, is the unique probability measure on $\{y \in \mathbb{R}^d : |y| = 1\}$ that is invariant under all vector isometries. For every $x \in \mathbb{R}^d$ and $r > 0$, we then let $\sigma_{x,r}(dy)$ be the image of $\sigma_1(dy)$ under the mapping $y \mapsto x + ry$.

Proposition 7.4 (Mean value property)

Suppose that u is harmonic on the domain D. Then, for every $x \in D$ and for every $r > 0$ such that $\overline{B(x,r)} \subset D$, we have

$$u(x) = \int \sigma_{x,r}(dy) u(dy).$$

Proof of Proposition 7.3 (cont)

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Proof of Proposition 7.4

First observe that, if $T_1 = \inf\{t \geq 0 : |B_t| = 1\}$, the distribution of B_{T_1} under $\mathbb{P}_0$ is invariant under all vector isometries of $\mathbb{R}^d$ (by the invariance properties of Brownian motion) and therefore this distribution is $\sigma_1(dy)$. By scaling and translation invariance, it follows that for every $x \in \mathbb{R}^d$ and $r > 0$, if $T_{x,r} = \inf\{t \geq 0 : |B_t - x| = r\}$, the distribution of $B_{T_{x,r}}$ under $\mathbb{P}_x$ is $\sigma_{x,r}$. However, Proposition 7.3 implies that, under the conditions in the proposition, we must have $u(x) = \mathbb{E}_x[u(B_{T_{x,r}})]$. The desired result follows.

We say that a (locally bounded and measurable) function u on D satisfies the mean value property if the conclusion of Proposition 7.4 holds. It turns out that this property characterizes harmonic functions.

Proof of Proposition 7.4

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We say that a (locally bounded and measurable) function u on D satisfies the mean value property if the conclusion of Proposition 7.4 holds. It turns out that this property characterizes harmonic functions.

Lemma 7.5

Let u be a locally bounded and measurable function on D that satisfies the mean value property. Then u is harmonic on D.

Proof of Lemma 7.5

Fix $r_0 > 0$ and let D' be the open subset of D consisting of all points whose distance to D^c is greater than r_0. It suffices to prove that u is twice continuously differentiable and $\Delta u = 0$ on D'. Let $h : \mathbb{R} \to \mathbb{R}_+$ be a C^∞ function with compact support contained in $(0, r_0)$ and not identically zero. Then, for every $x \in D'$ and every $r \in (0, r_0)$, we have

$$u(x) = \int \sigma_{x,r}(dy)u(dy).$$

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$$u(x) = \int \sigma_{x,r}(dy)u(dy).$$

Proof of Lemma 7.5 (cont)

Multiply both sides of this equality by $r^{d-1}h(r)$ and integrate with respect to Lebesgue measure dr on $(0, r_0)$. Using spherical coordinates, and agreeing for definiteness that $u = 0$ on D^c, we get, for every $x \in D'$,

$$cu(x) = \int_{\{|y|<r_0\}} dy h(|y|)u(x+y) = \int_{\mathbb{R}^d} dz h(|z-x|)u(z),$$

where $c > 0$ is a constant and we use the fact that $h(|z-x|) = 0$ if $|z-x| \geq r_0$. Since $z \mapsto h(|z|)$ is a C^∞ function, the convolution in the right-hand side of the last display also defines a C^∞ function on D'.

It remains to check that $\Delta u = 0$ on D'. To this end, we use a probabilistic argument. By applying Ito's formula to $u(B_t)$ under $\mathbb{P}_x$, we get, for $x \in D'$ and $r \in (0, r_0)$,

$$\mathbb{E}_x[u(B_{t\wedge T_{x,r}})] = u(x) + \frac{1}{2}\mathbb{E}_x\left[\int_0^{t\wedge T_{x,r}} ds \Delta u(B_s)\right].$$

If we $t \uparrow \infty$, noting that $\mathbb{E}_x[T_{x,r}] < \infty$, we get

$$\mathbb{E}_x[u(B_{T_{x,r}})] = u(x) + \frac{1}{2}\mathbb{E}_x\left[\int_0^{T_{x,r}} ds \Delta u(B_s)\right].$$

The mean value property just says that $\mathbb{E}_x[u(B_{T_{x,r}})] = u(x)$ and so we have

$$\mathbb{E}_x\left[\int_0^{T_{x,r}} ds \Delta u(B_s)\right] = 0.$$

Since this holds for any $r \in (0, r_0)$, it follows that $\Delta u(x) = 0$.

From now on, we assume that the domain D is bounded.

Definition 7.6 (Classical Dirichlet problem)

Let g be a continuous function on ∂D. A function $u : D \to \mathbb{R}$ solves the Dirichlet problem in D with boundary condition g, if u is harmonic on D and has boundary condition g, in the sense that, for every $y \in \partial D$, $\lim_{D \ni x \to y} u(x) = g(y)$.

If u solves the Dirichlet problem with boundary condition g, the function $\tilde{u}$ defined on $\overline{D}$ by $\tilde{u}(x) = u(x)$ if $x \in D$, and $\tilde{u}(x) = g(x)$ if $x \in \partial D$, is then continuous, hence bounded on $\overline{D}$.

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Proposition 7.7

Let D be a bounded domain, and write $T = \inf\{t \geq 0 : B_t \notin D\}$ for the exit time of Brownian motion from D.

(i) Let g be a continuous function on ∂D, and let u be a solution of the Dirichlet problem in D with boundary condition g. Then, for every $x \in D$,

$$u(x) = \mathbb{E}_x[g(B_T)].$$

(ii) Let g be a bounded measurable function on ∂D. Then the function

$$u(x) = \mathbb{E}_x[g(B_T)], \quad x \in D$$

is harmonic on D.

Remark

Assertion (i) implies that, if a solution to the Dirichlet problem with boundary condition g exists, then it is unique, which is also easy to prove using the mean value property.

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Remark

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Proof of Proposition 7.7

(i) Fix $x \in D$ and $\epsilon_0 > 0$ such that $\overline{B(x, \epsilon_0)} \subset D$. For every $\epsilon \in (0, \epsilon_0)$, let D_ϵ be the connected component containing x of the open set consisting of all points of D whose distance to D^c is greater than ϵ. If $T_\epsilon = \inf\{t \geq 0 : B_t \notin D_\epsilon\}$, Proposition 7.3 shows that

$$u(x) = \mathbb{E}_x[u(B_{T_\epsilon})].$$

Now observe that $T_\epsilon \uparrow T$ as $\epsilon \downarrow 0$ (if T' is the increasing limit of T_ϵ as $\epsilon \downarrow 0$, we have $T' \leq T$ and on the other hand $B_{T'} \in \partial D$ by the continuity of sample paths). Using dominated convergence, it follows that $\mathbb{E}_x[u(B_{T_\epsilon})]$ converges to $\mathbb{E}_x[u(B_T)]$ as $\epsilon \downarrow 0$.

Proof of Proposition 7.7 (cont)

(ii) By Lemma 7.5, it suffices to show that the function $u(x) = \mathbb{E}_x[g(B_T)]$ satisfies the mean value property. Recall the $T_{x,r} = \inf\{t \geq 0 : |B_t - x| = r\}$ for $x \in \mathbb{R}^d$ and $r > 0$. Fix $x \in D$ and $r > 0$ so that $\overline{B(x,r)} \subset D$. We now apply the strong Markov property: let $\Phi(w)$, for $w \in C(\mathbb{R}_+, \mathbb{R}^d)$ with $w(0) \in D$, be the value of g at the first exit point of w from D (we let $\Phi(w) = 0$ if w never exits D) and we observe that we have

$$g(B_T) = \Phi(B_t, t \geq 0) = \Phi(B_{T_{x,r}} + t, t \geq 0)$$

because the paths $B_t, t \geq 0)$ and $(B_{T_{x,r}} + t, t \geq 0)$ have the same exit point from D. It follows that

$$u(x) = \mathbb{E}_x[g(B_T)] = \mathbb{E}_x[\Phi(B_{T_{x,r}} + t, t \geq 0)]$$
$$= \mathbb{E}_x[\mathbb{E}_{B_{T_{x,r}}}[\Phi(B_t, t \geq 0)]] = \mathbb{E}_x[u(B_{T_{x,r}})]$$

Since the law of $B_{T_{x,r}}$ under $\mathbb{P}_x$ is $\sigma_{x,r}$, this gives the mean value property.