Math 562 Fall 2020

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Outline

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HW5 is due today at noon. Please submit your homework via the course Moodle page. Make sure that your HW is uploaded successfully.

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6.5.1 Jump Processes on a Finite State Space

In this subsection, we assume that the state space E is finite (and equipped with the discrete topology). Note that any cadlag function $f \in \mathbb{D}(E)$ must be of the following type: There exists a $t_1 \in (0, \infty]$ such that $f(t) = f(0)$ for every $t \in [0, t_1)$, then, if $t_1 < \infty$, there exists a $t_2 \in (t_1, \infty]$ such that $f(t) = f(t_1)$ for every $t \in [t_1, t_2)$, and so on.

Consider a Feller semigroup $(Q_t)_{t \geq 0}$ on E. By the remark of the end of Sect. 6.3, we can construct, on a probability space Ω equipped with a right-continuous filtration $(\mathcal{F}_t)_{t \in [0,\infty]}$, a collection $(\mathbb{P}_x)_{x \in E}$ of probability measures and a process $(X_t)_{t \geq 0}$ with cadlag sample paths such that, under $\mathbb{P}_x$, X is Markov with semigroup $(Q_t)_{t \geq 0}$ with respect to the filtration $(\mathcal{F}_t)$, and $\mathbb{P}(X_0 = x) = 1$. As before, we use $\mathbb{E}_x$ to denote expectation with respect to $\mathbb{P}_x$.

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Since the sample paths of X are cadlag, we know that, for every $\omega \in \Omega$, there exists a sequence

$$T_0(\omega) = 0 < T_1(\omega) \leq T_2(\omega) \leq T_3(\omega) \leq \cdots \leq \infty,$$

such that $X_t(\omega) = X_0(\omega)$ for every $t \in [0, T_1(\omega))$ and, for every integer $i \geq 1$, $T_i(\omega) < \infty$ implies that $T_i(\omega) < T_{i+1}(\omega)$, $X_{T_i(\omega)}(\omega) \neq X_{T_{i-1}(\omega)}(\omega)$ and $X_t(\omega) = X_{T_i(\omega)}(\omega)$ for every $t \in [T_i(\omega), T_{i+1}(\omega))$. Moreover, $T_n \uparrow \infty$ as $n \uparrow \infty$.

$T_0, T_1, T_2, \ldots$ are stopping times.

In the following lemma, we make the convention that an exponential variable with parameter 0 is equal to ∞ a.s.

Lemma 6.18

Let $x \in E$. There exists a number $q(x) \geq 0$ such that the random variable T_1 is exponentially distributed with parameter $q(x)$ under $\mathbb{P}_x$. Furthermore, if $q(x) > 0$, then T_1 and X_{T_1} are independent under $\mathbb{P}_x$.

Proof of Lemma 6.18

Let $s, t \geq 0$. We have

$$\mathbb{P}_x(T_1 > s + t) = \mathbb{E}_x[1_{\{T_1 > s\}} \Phi((X_{s+r})_{r \geq 0})],$$

where $\Phi(f) = 1_{\{f(r) = f(0), \forall r \in [0,t]\}}$ for $f \in \mathbb{D}(E)$. Using the simple Markov property (Theorem 6.16), we get

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Proof of Theorem 6.15 (cont)

$$\mathbb{P}_x(T_1 > s+t) = \mathbb{E}_x[1_{\{T_1>s\}}\mathbb{E}_{X_s}[\Phi((X_r)_{r\geq 0})]]$$
$$= \mathbb{E}_x[1_{\{T_1>s\}}\mathbb{P}_x(T_1 > t)] = \mathbb{P}_x(T_1 > s)\mathbb{P}_x(T_1 > t),$$

which implies that T_1 is exponentially distributed under $\mathbb{P}_x$.

Assume that $q(x) > 0$, so that $T_1 < \infty$ $\mathbb{P}_x$-a.s. Then, for every $t \geq 0$ and $y \in E$,

$$\mathbb{P}_x(T_1 > t, X_{T_1} = y) = \mathbb{E}_x[1_{\{T_1>t\}}\Psi((X_{t+r})_{r\geq 0})],$$

where for $f \in \mathbb{D}(E)$, $\Psi(f) = 0$ if f is constant, and otherwise $\Psi(f)$ $= 1_{\{\gamma_1(f)=y\}}$, if $\gamma_1(f)$ is the value of f after its first jump. We thus get

$$\mathbb{P}_x(T_1 > t, X_{T_1} = y) = \mathbb{E}_x[1_{\{T_1>t\}}\mathbb{E}_{X_t}[\Psi((X_r)_{r\geq 0})]]$$
$$= \mathbb{E}_x[1_{\{T_1>t\}}\mathbb{P}_x(X_{T_1} = y)] = \mathbb{P}(T_1 > t)\mathbb{P}_x(X_{T_1} = y)$$

which gives the desired independence.

Points x such that $q(x) = 0$ are absorbing states for the Markov process, in the sense that $\mathbb{P}_x(X_t = x, \forall t \geq 0) = 1$.

For every $x \in E$ such that $q(x) > 0$, and every $y \in E$, we define

$$\Pi(x, y) = \mathbb{P}_x(X_{T_1} = y).$$

Note that $\Pi(x, \cdot)$ is a probability measure on E, and $\Pi(x, x) = 0$.

Proposition 6.19

Let L denote the generator of $(Q_t)_{t \geq 0}$. Then $D(L) = C_0(E) = B(E)$, and, for every $\varphi \in B(E)$ and $x \in E$,

- if $q(x) = 0$, then $L\varphi(x) = 0$;
- if $q(x) > 0$,

$$L\varphi(x) = q(x) \sum_{E \ni y \neq x} \Pi(x, y)(\varphi(y) - \varphi(x)) = \sum_{y \in E} L(x, y)\varphi(y)$$

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Proposition 6.19 (cont)

where

$$L(x,y) = \begin{cases} q(x)\Pi(x,y), & \text{if } y \neq x, \\ -q(x), & \text{if } y = x. \end{cases}$$

Proof of Proposition 6.19

Let $\varphi \in B(E)$ and $x \in E$. If $q(x) = 0$, it is trivial that $Q_t\varphi(x) = \varphi(x)$ and so

$$\lim_{t \downarrow 0} \frac{Q_t\varphi(x) - \varphi(x)}{t} = 0.$$

Suppose now that $q(x) > 0$. We first note that, as $t \downarrow 0$,

$$\mathbb{P}_x(T_2 \leq t) = o(t^2). \tag{1}$$

Indeed, using the strong Markov property at T_1,

$$\mathbb{P}_x(T_2 \leq t) \leq \mathbb{P}_x(T_1 \leq t, T_2 \leq T_1 + t) = \mathbb{E}_x[1_{\{T_1 \leq t\}}\mathbb{P}_{X_{T_1}}(T_1 \leq t)]$$

and we can bound

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Proof of Proposition 6.19 (cont)

$$\mathbb{P}_{X_{T_1}}(T_1 \leq t) \leq \sup_{y \in E} \mathbb{P}_y(T_1 \leq t) \leq t \sup_{y \in E} q(y)$$

giving the desired result since we have also $\mathbb{P}_x(T_1 \leq t) \leq q(x)t$.

It follows from (1) that

$$Q_t\varphi(x) = \mathbb{E}_x[\varphi(X_t)] = \mathbb{E}_x[\varphi(X_t)1_{\{t<T_1\}}] + \mathbb{E}_x[\varphi(X_{T_1})1_{\{T_1 \leq t\}}] + o(t^2)$$
$$= \varphi(x)e^{-q(x)t} + (1 - e^{-q(x)t}) \sum_{E \ni y \neq x} \Pi(x,y)\varphi(y) + o(t^2),$$

using the independence of T_1 and X_{T_1}, and the definition of $\Pi(x,y)$. We conclude that, as $t \downarrow 0$,

$$\frac{Q_t\varphi(x) - \varphi(x)}{t} \to -q(x)\varphi(x) + q(x) \sum_{E \ni y \neq x} \Pi(x,y)\varphi(y),$$

and this completes the proof.

In particular, taking $\varphi(y) = 1_{\{y\}}$, we have if $y \neq x$,

$$L(x, y) = \frac{d}{dt}\mathbb{P}_y(X_t = y)|_{t=0}$$

so that $L(x, y)$ can be interpreted as the instantaneous rate of transition from x to y.

The next proposition provides a complete description of the sample paths of X under $\mathbb{P}_x$. For the sake of simplicity, we assume that there are no absorbing states, but one can easily extend the statement to the general case.

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Proposition 6.20

We assume that $q(y) > 0$ for every $y \in E$. Let $x \in E$. Then $\mathbb{P}_x$-a.s., the jump times $T_1 < T_2 < T_3 < \cdots$ are all finite and the sequence $X_0, X_{T_1}, X_{T_2}, \ldots$ is, under $\mathbb{P}_x$, a discrete Markov chain with transition kernel Π started at x. Furthermore, conditionally on $(X_0, X_{T_1}, X_{T_2}, \ldots)$, the random variables $T_1 - T_0, T_2 - T_1, \ldots$ are independent and, for every integer $i \geq 0$, the conditional distribution of $T_{i+1} - T_i$ is exponential with parameter $q(X_{T_i})$.

Proof of Proposition 6.20

An application of the strong Markov property shows that all stopping times $T_1, T_2, \ldots$ are finite $\mathbb{P}_x$-a.s. Then, let $y, z \in E$ and $f_1, f_2 \in B(\mathbb{R}_+)$. By the strong Markov property at T_1,

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Proof of Proposition 6.20 (cont)

$$\begin{aligned}
&\mathbb{E}_x[1_{\{X_{T_1}=y\}} f_1(T_1) 1_{\{X_{T_2}=z\}} f_2(T_2 - T_1)] \\
&= \mathbb{E}_x[1_{\{X_{T_1}=y\}} f_1(T_1) \mathbb{E}_{X_{T_1}}[1_{\{X_{T_1}=z\}} f_2(T_1)]] \\
&= \Pi(x,y)\Pi(y,z) \int_0^\infty ds_1 e^{-q(x)s_1} f_1(s_1) \int_0^\infty ds_2 e^{-q(y)s_2} f_2(s_2).
\end{aligned}$$

By induction, we get for every $y_1, \ldots, y_p \in E$ and $f_1, \ldots, f_p \in B(\mathbb{R}_+)$,

$$\begin{aligned}
&\mathbb{E}_x[1_{\{X_{T_1}=y_1\}} 1_{\{X_{T_2}=y_2\}} \cdots 1_{\{X_{T_p}=y_p\}} f_1(T_1) f_2(T_2 - T_1) \cdots f_p(T_p - T_{p-1})] \\
&= \Pi(x,y_1)\Pi(y_1,y_2) \cdots \Pi(y_{p-1},y_p) \prod_{i=1}^{p} \left(\int_0^\infty ds e^{-q(y_{i-1})s} f_i(s) \right)
\end{aligned}$$

where $y_0 = x$ by convention. The various assertions of the proposition follow.

Jump processes play an important role in various models of applied probability, in particular in reliability and in queueing theory. In such applications, one usually starts from the transition rates of the process. It is thus important to know whether, given a collection $(q(x))_{x\in E}$ of nonnegative numbers and, for every $x \in E$ with $q(x) > 0$, a probability measure $\Pi(x, \cdot)$ on E such that $\Pi(x, x) = 0$, there exists a corresponding Feller semigroup $(Q_t)_{t\geq 0}$ and therefore an associated Markov process. The answer to this question is yes, and one can give two different arguments:

- Probabilistic method. Use the description of Proposition 6.20 (or its extension to the case where there are absorbing states) to construct the process $(X_t)_{t\geq 0}$ starting from any $x \in E$, and thus the semigroup $(Q_t)_{t\geq 0}$ via the formula $Q_t\varphi(x) = \mathbb{E}_x[\varphi(X_t)]$.

- Analytic method. Define the generator L via the formulas of Proposition 6.19, and observe that the semigroup $(Q_t)_{t\geq 0}$, if it exists, must solve the differential equation

$$\frac{d}{dt}Q_t(x,y) = Q_t L(x,y)$$

by Proposition 6.11. This leads to

$$Q_t = \exp(tL)$$

in the sense of the exponential of matrices. Since $\lambda Id + L$ has non-negative entries if $\lambda > 0$ is large enough, one immediately gets that Q_t has nonnegative entries. Writing $\mathbf{1}$ for the vector $(1, 1, \ldots, 1)$, the property $L\mathbf{1} = 0$ gives $Q_t\mathbf{1} = \mathbf{1}$ so that $(Q_t(x,\cdot)_{x\in E}$ defines a transition kernel. Finally, the property $\exp((s+t)L) = \exp(sL)\exp(tL)$ the Chapman-Kolmogorov property, and we get that $(Q_t)_{t\geq 0}$ is a transition semigroup on E, whose Feller property is also immediate.

Many of the preceding results can be extended to Feller Markov processes on a countable state space E. Note, however, that certain difficulties arise in the question of the existence of a process with given transition rates. In fact, starting from the probabilistic description of Proposition 6.20, one needs to avoid the possibility of an accumulation of jumps in a finite time interval, which may occur if the rates $(q(y))_{y \in E}$ are unbounded – of course this problem does not occur when E is finite.

6.5.2 Lévy processes

Consider an $\mathbb{R}^d$-valued process $(Y_t)_{t\geq 0}$ satisfying the following three assumptions:

(i) $Y_0 = 0$ a.s.

(ii) For every $0 \leq s \leq t$, $Y_t - Y_s$ is independent of $(Y_r : 0 \leq r \leq s)$ and has the same law as Y_{t-s}.

(iii) Y_t converges in probability to 0 when $t \downarrow 0$.

Two special cases are $\mathbb{R}^d$-valued Brownian motion started from the origin and the hitting times $(T_a)_{a\geq 0}$ of a 1-dim Brownian motion.

Note that we do not assume that sample paths of Y are cadlag, but only the weaker regularity assumption (iii). The theory of previous sections will allow us to find a modification of Y with cadlag sample paths.

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For every $t \geq 0$, we denote the law of Y_t by $Q_t(0, dy)$, and, for every $x \in \mathbb{R}^d$, we let $Q_t(x, dy)$ be the image of $Q_t(0, dy)$ under the translation $y \mapsto x + y$.

Proposition 6.21

The collection $(Q_t)_{t \geq 0}$ is a Feller semigroup on $\mathbb{R}^d$. Furthermore, $(Y_t)_{t \geq 0}$ is a Markov process with semigroup $(Q_t)_{t \geq 0}$.

Proof of Proposition 6.21

Let us first show that $(Q_t)_{t \geq 0}$ is a transition semigroup. Let $\varphi \in B(\mathbb{R}^d)$, $s, t \geq 0$ and $x \in \mathbb{R}^d$. Property (ii) shows that the law of $(Y_t, Y_{t+s} - Y_t)$ is the product probability measure $Q_t(0, \cdot) \otimes Q_s(0, \cdot)$. Hence

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Proof of Proposition 6.21 (cont)

$$\int Q_t(x,dy)\int Q_s(y,dz)\varphi(z) = \int Q_t(0,dy)\int Q_s(0,dz)\varphi(x+y+z)$$

$$= \mathbb{E}[\varphi(x+Y_t+(Y_{t+s}-Y_t))] = \mathbb{E}[\varphi(x+Y_{t+s})] = \int Q_{t+s}(x,dz)\varphi(z)$$

giving the Chapman-Kolmogorov relation.We should also verify the measurability of the mapping $(t,x)\mapsto Q_t(x,A)$, but this will follow from the stronger continuity properties that we will establish in order to verify the Feller property.

Let us start with the first property of the definition of a Feller semigroup. If $\varphi\in C_0(\mathbb{R}^d)$, the mapping

$$x\mapsto Q_t\varphi(x) = \mathbb{E}[\varphi(x+Y_t)]$$

is continuous by dominated convergence, and, again by dominated convergence, we have, as $|x|\to\infty$,

Proof of Proposition 6.21 (cont)

$$\mathbb{E}[\varphi(x + Y_t)] \to 0$$

showing that $Q_t\varphi \in C_0(\mathbb{R}^d)$. Then, as $t \downarrow 0$,

$$Q_t\varphi(x) = \mathbb{E}[\varphi(x + Y_t)] \to \varphi(x)$$

thanks to property (iii). The uniform continuity of φ even shows that the latter convergence is uniform in x. This completes the proof of the first assertion of the proposition. To get the second one, we write, for every $s, t \geq 0$ and $\varphi \in B(\mathbb{R}^d)$,

$$\mathbb{E}[\varphi(Y_{s+t}|Y_r : 0 \leq r \leq s] = \mathbb{E}[\varphi(Y_s + (Y_{s+t} - Y_s))|Y_r : 0 \leq r \leq s]$$
$$= \int \varphi(Y_s + y)Q_t(0, dy) = \int \varphi(y)Q_t(Y_s, dy),$$

using property (ii) and the definition of $Q_t(0, \cdot)$ in the second equality.

It then follows from Theorem 6.15 that there exists a modification of $(Y_t)_{t\geq 0}$ with cadlag sample paths. Obviously this modification still satisfies (i) and (ii).

A Lévy process in $\mathbb{R}^d$ is a process satisfying properties (i) and (ii) above, and having cadlag sample paths (which implies (iii)).