Math 562 Fall 2020

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Outline

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HW5 is due Friday Nov. 6 at noon. Please submit your homework via the course Moodle page. Make sure that your HW is uploaded successfully.

Outline

We consider a Feller semigroup $(Q_t)_{t\geq 0}$ on a locally compact separable metric space E.

Theorem 6.15

Let $(X_t)_{t\geq 0}$ be a Markov process with semigroup $(Q_t)_{t\geq 0}$, with respect to the filtration $(\mathcal{F}_t)_{t\in[0,\infty]}$. Set $\widetilde{\mathcal{F}}_\infty = \mathcal{F}_\infty$, and, for every $t \geq 0$,

$$\widetilde{\mathcal{F}}_t = \mathcal{F}_{t+} \vee \sigma(\mathcal{N})$$

where $\mathcal{N}$ is the family of all $\mathcal{F}_\infty$-measurable sets of zero probability. Then, the process $(X_t)_{t\geq 0}$ has a cadlag modification $(\widetilde{X}_t)_{t\geq 0}$, which is adapted to the filtration $(\widetilde{\mathcal{F}}_t)$. Moreover, $(\widetilde{X}_t)_{t\geq 0}$ is a Markov process with semigroup $(Q_t)_{t\geq 0}$, with respect to the filtration $(\widetilde{\mathcal{F}}_t)_{t\in[0,\infty]}$.

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where $\mathcal{N}$ is the family of all $\mathcal{F}_\infty$-measurable sets of zero probability. Then, the process $(X_t)_{t\geq 0}$ has a cadlag modification $(\widetilde{X}_t)_{t\geq 0}$, which is adapted to the filtration $(\widetilde{\mathcal{F}}_t)$. Moreover, $(\widetilde{X}_t)_{t\geq 0}$ is a Markov process with semigroup $(Q_t)_{t\geq 0}$, with respect to the filtration $(\widetilde{\mathcal{F}}_t)_{t\in[0,\infty]}$.

Proof of Theorem 6.15 (cont)

Recall that $E_\Delta = E \cup \{\Delta\}$ is the one-point compactification of E. Every function $f \in C_0(E)$ is extended to a continuous function on E_Δ by setting $f(\Delta) = 0$.

Let $(f_n)_{n\geq 0}$ be a sequence in $C_0^+(E)$ which separates the points of E_Δ. Then

$$\mathcal{H} = \{R_p f_n : p \geq 1, n \geq 0\}$$

is also a countable subset of $C_0^+(E)$ which separates the points of E_Δ. If $h \in \mathcal{H}$, then there exists an integer $p \geq 1$ such that $e^{-pt}h(X_t)$ is a supermartingale. Let D be a countable dense subset of $\mathbb{R}_+$. Then Theorem 3.17 (i) shows that the limits

$$\lim_{D \ni s \downarrow\downarrow t} h(X_s), \qquad \lim_{D \ni s \uparrow\uparrow t} h(X_s)$$

exist simultaneously for every $t \in \mathbb{R}_+$ (the second one only for $t > 0$) outside an $\mathcal{F}_\infty$-measurable event N_h of zero probability.

Proof of Theorem 6.15 (cont)

We then set

$$N = \cup_{h \in \mathcal{H}} N_h,$$

then we still have $N \in \mathcal{N}$. Then if $\omega \notin N$, the limits

$$\lim_{D \ni s \downarrow\downarrow t} X_s, \qquad \lim_{D \ni s \uparrow\uparrow t} X_s$$

exist simultaneously for every $t \in \mathbb{R}_+$ (the second one only for $t > 0$) in E_Δ. We can then set, for every $\omega \in \Omega \setminus N$ and every $t \geq 0$,

$$\widetilde{X}_t(\omega) = \lim_{D \ni s \downarrow\downarrow t} X_s(\omega).$$

If $\omega \in N$, we set $\widetilde{X}_t(\omega) = x_0$ for every $t \geq 0$, where x_0 is a fixed point in E. Then, for every $t \geq 0$, $\widetilde{X}_t$ is an $\widetilde{\mathcal{F}}_t$-measurable random variable with values in E_Δ. Furthermore, for every $\omega \in \Omega$, $t \mapsto \widetilde{X}_t(\omega)$, viewed as a mapping with values in E_Δ, is cadlag by Lemma 3.16. We have already showed that $(\widetilde{X}_t)$ is a modification of (X_t).

Proof of Theorem 6.15 (cont)

Let us now show that $(\widetilde{X}_t)_{t\geq 0}$ is a Markov process with semigroup $(Q_t)_{t\geq 0}$ with respect to the filtration $(\widetilde{\mathcal{F}}_t)$. It suffices to prove that, for every $s \geq 0$, $t > 0$, $A \in \widetilde{\mathcal{F}}_s$ and $f \in C_0(E)$,

$$\mathbb{E}[1_A f(\widetilde{X}_{s+t})] = \mathbb{E}[1_A Q_t f(\widetilde{X}_s)].$$

Since $\widetilde{X}_s = X_s$ a.s. and $\widetilde{X}_{s+t} = X_{s+t}$ a.s, this is equivalent to proving that

$$\mathbb{E}[1_A f(X_{s+t})] = \mathbb{E}[1_A Q_t f(X_s)].$$

Because A is equal a.s. to an element of $\mathcal{F}_{s+}$, we may assume that $A \in \mathcal{F}_{s+}$. Let (s_n) be a sequence in D that decreases to s, so that $A \in \mathcal{F}_{s_n}$ for every n. Then, as soon as $s_n \leq s + t$, we have

$$\mathbb{E}[1_A f(X_{s+t})] = \mathbb{E}[1_A \mathbb{E}[f(X_{s+t})|\mathcal{F}_{s_n}]] = \mathbb{E}[1_A Q_{s+t-s_n} f(X_{s_n})].$$

Proof of Theorem 6.15 (cont)

But $Q_{s+t-s_n}f$ converges (uniformly) to $Q_t f$ by properties of Feller semigroups, and since $X_{s_n} = \widetilde{X}_{s_n}$ a.s. we also know that X_{s_n} converges a.s. to $\widetilde{X}_s = X_s$. We thus obtain the desired result by letting n tend to ∞.

It remains to show that the sample paths $t \mapsto \widetilde{X}_t(\omega)$ are cadlag as E-valued mappings, and not only as E_Δ-valued mappings (we already know that, for every fixed $t \geq 0$, $X_t(\omega) = \widetilde{X}_t(\omega)$ a.s. is in E with probability one, but this does not imply that the sample paths, and their left-limits, remain in E). Fix a function $g \in C_0^+(E)$ such that $g(x) > 0$ for every $x \in E$. The function $h = R_1 g$ then satisfies the same property. Set, for every $t \geq 0$,

$$Y_t = e^{-t}h(\widetilde{X}_t).$$

Then Lemma 6.6 shows that $(Y_t)_{t\geq 0}$ is a non-negative supermartingale with respect to the filtration $(\widetilde{\mathcal{F}}_t)$. Additionally, we know that the sample paths of $(Y_t)_{t\geq 0}$ are cadlag (recall that $h(\Delta) = 0$ by convention).

Proof of Theorem 6.15 (cont)

For every integer $n \geq 1$, set

$$T_{(n)} = \inf\{t \geq 0 : Y_t < \frac{1}{n}\}.$$

Then $T_{(n)}$ is a stopping time of the filtration $(\widetilde{\mathcal{F}}_t)$. Consequently,

$$T = \lim_{n \to \infty} \uparrow T_{(n)}$$

is a stopping time. The desired result will follow if we can verify that $\mathbb{P}(T < \infty) = 0$. Indeed, it is clear that, for every $t \in [0, T_{(n)})$, $\widetilde{X}_t \in E$ and $\widetilde{X}_{t-} \in E$ and we may redefine $\widetilde{X}_t(\omega) = x_0$ (for every $t \geq 0$) for all ω belonging to $\{T < \infty\} \in \mathcal{N}$.

Proof of Theorem 6.15 (cont)

To show $\mathbb{P}(T < \infty) = 0$, we apply Theorem 3.25 and the subsequent remark to $Z = Y$ and $U = T_{(n)}$ and $V = T + q$, where $q > 0$ s a rational number. We get

$$\mathbb{E}[Y_{T+q} 1_{\{T<\infty\}}] \leq \mathbb{E}[Y_{T_{(n)}} 1_{\{T_{(n)}<\infty\}}] \leq \frac{1}{n}.$$

By letting n tend to ∞, we get

$$\mathbb{E}[Y_{T+q} 1_{\{T<\infty\}}] = 0,$$

hence $Y_{T+q} = 0$ a.s. on $\{T < \infty\}$. By the right-continuity of sample paths of Y, we conclude that $Y_t = 0$, for every $t \in [T, \infty)$, a.s. on $\{T < \infty\}$. But we also know that, for every integer $k \geq 0$, $Y_k = e^{-k} h(\widetilde{X}_k) > 0$ a.s. since $\widetilde{X}_k \in E$. This suffices to get $\mathbb{P}(T < \infty) = 0$.

Remark

The proof above applies with minor modifications to the different setting where we are given the process $(X_t)_{t\geq 0}$ together with a collection $(\mathbb{P}_x)_{x\in E}$ of probability measures such that, under $\mathbb{P}_x$, $(X_t)_{t\geq 0}$ is a Markov process with semigroup $(Q_t)_{t\geq 0}$, with respect to a filtration $(\mathcal{F}_t)_{t\in[0,\infty]}$, and $\mathbb{P}(X_0 = x) = 1$. In this setting, we can define the filtration $(\widetilde{\mathcal{F}}_t)_{t\in[0,\infty]}$ by

$$\widetilde{\mathcal{F}}_t = \mathcal{F}_{t+} \vee \sigma(\mathcal{N}')$$

where $\mathcal{N}'$ is the family of all the $\mathcal{F}_\infty$-measurable sets that have zero $\mathbb{P}_x$ probability for every $x \in E$. By the same arguments as in the preceding proof, we can then construct an $(\widetilde{\mathcal{F}}_t)$-adapted process $(\widetilde{X}_t)$ with cadlag sample paths, such that, for every $x \in E$,

$$\mathbb{P}_x(\widetilde{X}_t = X_t) = 1, \quad \forall t \geq 0,$$

and $(\widetilde{X}_t)_{t\geq 0}$ is under $\mathbb{P}_x$ a Markov process with semigroup (Q_t), with respect to the filtration $(\widetilde{\mathcal{F}}_t)_{t\in[0,\infty]}$, such that $\mathbb{P}(\widetilde{X}_0 = x) = 1$.

Outline

In the first part of this section, we return to the general setting of Sect. 6.1 above, where $(Q_t)_{t\geq 0}$ is a (not necessarily Feller) transition semigroup on E. We assume here that E is a metric space (equipped with its Borel σ-field), and moreover that, for every $x \in E$, one can construct a Markov process $(X_t^x)_{t\geq 0}$ with semigroup $(Q_t)_{t\geq 0}$ such that $X_0^x = x$ a.s. and the sample paths of X are cadlag. In the case of a Feller semigroup, the existence of such a process follows from Corollary 6.4 and Theorem 6.15.

The space of all cadlag paths $f : \mathbb{R}_+ \to E$ is denoted by $\mathbb{D}(E)$. This space is equipped with the σ-field $\mathcal{D}$ generated by the coordinate mappings $f \mapsto f(t)$. For every $x \in E$, we write $\mathbb{P}_x$ for the probability measure on $\mathbb{D}(E)$ which is the law of the random path $(X_t^x)_{t\geq 0}$. Notice that $\mathbb{P}_x$ does not depend on the choice of X^x, nor of the probability space where X^x is defined. This follows from the fact that the finite dimensional marginals of a Markov process are determined by its semigroup and initial distribution.

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We first give a version of the (simple) Markov property, which is a simple extension of the definition of a Markov process. We use the notation $\mathbb{E}_x$ for the expectation under $\mathbb{P}_x$.

Theorem 6.16 (Simple Markov property)

Let $(Y_t)_{t\geq 0}$ be a Markov process with semigroup $(Q_t)_{t\geq 0}$, with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. We assume that the sample paths of Y are cadlag. Let $s \geq 0$ and let $\Phi : \mathbb{D}(E) \to \mathbb{R}_+$ be a measurable function. Then,

$$\mathbb{E}[\Phi((Y_{s+t})_{t\geq 0})|\mathcal{F}_s] = \mathbb{E}_{Y_s}[\Phi].$$

Remark

The right-hand side of the last display is the composition of Y_s and of the mapping $y \mapsto \mathbb{E}_y[\Phi]$. To see that the latter mapping is measurable, it is enough to consider the case where $\Phi = 1_A$, $A \in \mathcal{D}$. When A only depends on a finite number of coordinates, there is an explicit formula, and an application of the monotone class lemma completes the argument.

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Proof of Theorem 6.16

As in the preceding remark, it suffices to consider the case where $\Phi = 1_A$ and

$$A = \{f \in \mathbb{D}(E) : f(t_1) \in B_1, \dots, f(t_p) \in B_p\},$$

where $0 \le t_1 < t_2 < \cdots < t_p$ and $B_1, \dots, B_p \in \mathcal{B}(E)$. In this case, we need to verify that

$$\begin{aligned}
&\mathbb{P}(Y_{s+t_1} \in B_1, \dots, Y_{s+t_p} \in B_p | \mathcal{F}_s) \\
&= \int_{B_1} Q_{t_1}(Y_s, dx_1) \int_{B_2} Q_{t_2 - t_1}(x_1, dx_2) \cdots \int_{B_p} Q_{t_p - t_{p-1}}(x_{p-1}, dx_p).
\end{aligned}$$

We in fact prove more generally that, if $\varphi_1, \dots, \varphi_p \in \mathcal{B}(E)$,

$$\begin{aligned}
&\mathbb{E}[\varphi_1(Y_{s+t_1}) \cdots \varphi_p(Y_{s+t_p}) | \mathcal{F}_s] \\
&= \int Q_{t_1}(Y_s, dx_1)\varphi_1(x_1) \int Q_{t_2 - t_1}(x_1, dx_2)\varphi_2(x_2) \cdots \int Q_{t_p - t_{p-1}}(x_{p-1}, dx_p)\varphi_p(x
\end{aligned}$$

Proof of Theorem 6.16 (cont)

If $p = 1$, this is the definition of a Markov process. We then argue by induction, writing

$$\begin{aligned}
&\mathbb{E}[\varphi_1(Y_{s+t_1})\cdots\varphi_p(Y_{s+t_p})|\mathcal{F}_s] \\
&= \mathbb{E}[\varphi_1(Y_{s+t_1})\cdots\varphi_{p-1}(Y_{s+t_{p-1}})\mathbb{E}[\varphi_p(Y_{s+t_p})|\mathcal{F}_{s+t_{p-1}}]|\mathcal{F}_s] \\
&= \mathbb{E}[\varphi_1(Y_{s+t_1})\cdots\varphi_{p-1}(Y_{s+t_{p-1}})Q_{t_p-t_{p-1}}(Y_{s+t_{p-1}})|\mathcal{F}_s]
\end{aligned}$$

and the desired result easily follows.

We now turn to the strong Markov property.

Proof of Theorem 6.16 (cont)

If $p = 1$, this is the definition of a Markov process. We then argue by induction, writing

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&\mathbb{E}[\varphi_1(Y_{s+t_1})\cdots\varphi_p(Y_{s+t_p})|\mathcal{F}_s] \\
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&= \mathbb{E}[\varphi_1(Y_{s+t_1})\cdots\varphi_{p-1}(Y_{s+t_{p-1}})Q_{t_p-t_{p-1}}(Y_{s+t_{p-1}})|\mathcal{F}_s]
\end{aligned}$$

and the desired result easily follows.

We now turn to the strong Markov property.

Theorem 6.17 (Strong Markov property)

Retain the assumptions of the previous theorem, and suppose in addition that $(Q_t)_{t\geq 0}$ is a Feller semigroup (in particular, E is assumed to be a locally compact separable metric space). Let T be a stopping time of the filtration $(\mathcal{F}_{t+})$, and let $\Phi : \mathbb{D}(E) \to \mathbb{R}_+$ be a measurable function. Then, for every $x \in E$,

$$\mathbb{E}[1_{\{T<\infty\}}\Phi((Y_{T+t})_{t\geq 0})|\mathcal{F}_T] = 1_{\{T<\infty\}}\mathbb{E}_{Y_T}[\Phi].$$

Remark

We allow T to be a stopping time of $(\mathcal{F}_{t+})$, which is slightly more general than saying that T is a stopping time of the filtration $(\mathcal{F}_t)$.

Theorem 6.17 (Strong Markov property)

Retain the assumptions of the previous theorem, and suppose in addition that $(Q_t)_{t\geq 0}$ is a Feller semigroup (in particular, E is assumed to be a locally compact separable metric space). Let T be a stopping time of the filtration $(\mathcal{F}_{t+})$, and let $\Phi : \mathbb{D}(E) \to \mathbb{R}_+$ be a measurable function. Then, for every $x \in E$,

$$\mathbb{E}[1_{\{T<\infty\}}\Phi((Y_{T+t})_{t\geq 0})|\mathcal{F}_T] = 1_{\{T<\infty\}}\mathbb{E}_{Y_T}[\Phi].$$

Remark

We allow T to be a stopping time of $(\mathcal{F}_{t+})$, which is slightly more general than saying that T is a stopping time of the filtration $(\mathcal{F}_t)$.

Proof of Theorem 6.17

We first observe that the right-hand side of the last display is $\mathcal{F}_T$-measurable, because $\{T < \infty\} \ni \omega \mapsto Y_T(\omega)$ is $\mathcal{F}_T$-measurable and the function $y \mapsto \mathbb{E}_y[\Phi]$ is measurable. It is then enough to show that, for $A \in \mathcal{F}_T$ fixed,

$$\mathbb{E}[1_{A\cap\{T<\infty\}}\Phi((Y_{T+t})_{t\geq 0})] = \mathbb{E}[1_{A\cap\{T<\infty\}}\mathbb{E}_{Y_T}[\Phi]].$$

As above, we can restrict our attention to the case where

$$\Phi(f) = \varphi_1(f(t_1))\cdots\varphi_p(f(t_p))$$

where $0 \leq t_1 < t_2 < \cdots < t_p$ and $\varphi_1, \ldots, \varphi_p \in \mathcal{B}(E)$. It is in fact enough to take $p = 1$.

Proof of Theorem 6.17 (cont)

If the desired result holds in this case, we can argue by induction, writing

$$\begin{aligned}
&\mathbb{E}[1_{A\cap\{T<\infty\}}\varphi_1(Y_{T+t_1})\cdots\varphi_p(Y_{T+t_p})]\\
&=E[1_{A\cap\{T<\infty\}}\varphi_1(Y_{T+t_1})\cdots\varphi_{p-1}(Y_{T+t_{p-1}})\mathbb{E}[\varphi_p(Y_{T+t_p})|\mathcal{F}_{T+t_{p-1}}]]\\
&=E[1_{A\cap\{T<\infty\}}\varphi_1(Y_{T+t_1})\cdots\varphi_{p-1}(Y_{T+t_{p-1}})Q_{t_p-t_{p-1}}\varphi_p(Y_{T+t_{p-1}})]
\end{aligned}$$

We thus fix $t\geq 0$ and $\varphi\in\mathcal{B}(E)$ and we aim to prove that

$$\mathbb{E}[1_{A\cap\{T<\infty\}}\varphi(Y_{T+t})]=\mathbb{E}[1_{A\cap\{T<\infty\}}Q_T\varphi(Y_T)].$$

We may assume that $\varphi\in C_0(E)$ (a finite measure on E is determined by its values against functions of $C_0(E)$).

Proof of Theorem 6.17 (cont)

On the event $\{T < \infty\}$, write $[T]_n$ for the smallest real number of the form $i2^{-n}$, $i \in \mathbb{Z}_+$, which is strictly greater than T. Then,

$$\begin{aligned}
\mathbb{E}[1_{A\cap\{T<\infty\}}\varphi(Y_{T+t})] &= \lim_{n\to\infty} \mathbb{E}[1_{A\cap\{T<\infty\}}\varphi(Y_{[T]_n+t})] \\
&= \lim_{n\to\infty} \sum_{i=1}^{\infty} \mathbb{E}[1_{A\cap\{(i-1)2^{-n}\le T<i2^{-n}\}}\varphi(Y_{i2^{-n}+t})] \\
&= \lim_{n\to\infty} \sum_{i=1}^{\infty} \mathbb{E}[1_{A\cap\{(i-1)2^{-n}\le T<i2^{-n}\}}Q_t\varphi(Y_{i2^{-n}})] \\
&= \lim_{n\to\infty} E[1_{A\cap\{T<\infty\}}Q_t\varphi(Y_{[T]_n})] = E[1_{A\cap\{T<\infty\}}Q_t\varphi(Y_T)]
\end{aligned}$$

giving the desired result. In the first (and in the last) equality, we use the right continuity of sample paths. In the third equality, we observe that the event

$$A \cap \{(i-1)2^{-n} \le T < i2^{-n}\}$$

belongs to $\mathcal{F}_{i2^n}$ because $A \in \mathcal{F}_T$ and T is a stopping time of the filtration $(\mathcal{F}_{t+})$ (use Proposition 3.6).

Proof of Theorem 6.17 (cont)

Finally, and this is the key point, in the last equality we also use the fact that $Q_t\varphi$ is continuous, since $\varphi \in C_0(E)$ and the semigroup is Feller.