Math 562 Fall 2020

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Outline

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1. General Info
2. 6.2 Feller Semigroups
3. 6.3 The Regularity of Sample Paths

HW5 is due Friday Nov. 6 at noon. Please submit your homework via the course Moodle page. Make sure that your HW is uploaded successfully.

Outline

Example

It is easy to verify that the semigroup $(Q_t)_{t\geq 0}$ of a 1-dim Brownian motion is Feller. Let us compute its generator. We have seen that, for every $\lambda > 0$ and $f \in C_0(\mathbb{R})$,

$$R_\lambda f(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\lambda}} \exp(-\sqrt{2\lambda}|y-x|) f(y) dy.$$

If $h \in D(L)$, then there exists an $f \in C_0(\mathbb{R})$ such that $h = R_\lambda f$. Taking $\lambda = \frac{1}{2}$, we have

$$h(x) = \int_{-\infty}^{\infty} \exp(-|y-x|) f(y) dy.$$

Differentiating, we get that h is differentiable on $\mathbb{R}$, and

$$h'(x) = \int_{-\infty}^{\infty} \operatorname{sgn}(y-x) \exp(-|y-x|) f(y) dy$$

with $\operatorname{sgn}(z) = 1_{\{z>0\}} - 1_{\{z\leq 0\}}$.

We now show that h' is differentiable on $\mathbb{R}$. Let $x_0 \in \mathbb{R}$. For $x > x_0$,

$$\begin{aligned}
&h'(x) - h'(x_0)\\
&= \int_{-\infty}^{\infty} (\operatorname{sgn}(y-x)\exp(-|y-x|) - \operatorname{sgn}(y-x_0)\exp(-|y-x_0|)) f(y)dy\\
&= \int_{x_0}^{x} (-\exp(-|y-x|) - \exp(-|y-x_0|)) f(y)dy\\
&\quad + \int_{\mathbb{R}\setminus[x_0,x]} \operatorname{sgn}(y-x_0)(\exp(-|y-x|) - \exp(-|y-x_0|)) f(y)dy.
\end{aligned}$$

It follows that, as $x \downarrow x_0$,

$$\frac{h'(x) - h'(x_0)}{x - x_0} = -2f(x_0) + h(x_0).$$

We get the same limit when $x \uparrow x_0$, and we thus obtain that h is twice differentiable, and $h'' = -2f + h$.

On the other hand, since $h = R_{1/2}f$, Proposition 6.12 shows that

$$\left(\frac{1}{2} - L\right) h = f$$

hence $Lh = -f + \frac{1}{2}h = \frac{1}{2}h''$.

Summarizing, we have obtained that

$$D(L) \subset \{h \in C^2(\mathbb{R}) : h, h'' \in C_0(\mathbb{R})\}$$

and that, if $h \in D(L)$, we have $Lh = \frac{1}{2}h''$.

In fact, the preceding inclusion is an equality. To see this, we may argue in the following way. If $g \in C^2(\mathbb{R})$ is such that g and g'' are in $C_0(\mathbb{R})$, we set $f = \frac{1}{2}(g - g'') \in C_0(\mathbb{R})$, then $h = R_{1/2}f \in D(L)$. By the preceding argument, h is twice differentiable and $h'' = -2f + h$. It follows that $(h - g)'' = h - g$. Since the $h - g \in C_0(\mathbb{R})$, it must vanish identically and we get $g = h \in D(L)$.

In general, it is very difficult to determine the exact domain of the generator. The following theorem often allows one to identify elements of this domain using martingales associated with the Markov process with semigroup $(Q_t)_{t\geq 0}$.

We consider again a general Feller semigroup $(Q_t)_{t\geq 0}$. We assume that on some probability space, we are given, for every $x \in E$, a process $(X_t^x)_{t\geq 0}$ which is Markov with semigroup $(Q_t)_{t\geq 0}$, with respect to a filtration $(\mathcal{F}_t)$, and such that $\mathbb{P}(X_0^x = x) = 1$. To make sense of the integrals that will appear below, we also assume that the sample paths of $(X_t^x)_{t\geq 0}$ are cadlag (we will see in the next section that this assumption is not restrictive).

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Theorem 6.14

Let $h, g \in C_0(E)$. The following two conditions are equivalent:

(i) $h \in D(L)$ and $Lh = g$.

(ii) For every $x \in E$, the process

$$h(X_t^x) - \int_0^t g(X_s^x)ds$$

is a martingale, with respect to the filtration $(\mathcal{F}_t)$.

Proof of Theorem 6.14

(i)$\Rightarrow$ (ii) Let $h \in D(L)$ and $g = Lh$. By Proposition 6.11, we have then, for every $s \geq 0$,

$$Q_s h = h + \int_0^s Q_r g dr.$$

It follows that, for $t \geq 0$ and $s \geq 0$,

$$\mathbb{E}[h(X_{t+s}^x)|\mathcal{F}_t] = Q_s h(X_t^x) = h(X_t^x) + \int_0^s Q_r g(X_t^x)dr.$$

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Proof of Theorem 6.14 (cont)

On the other hand,

$$\mathbb{E}\left[\int_t^{t+s} g(X_r^x)dr|\mathcal{F}_t\right] = \int_t^{t+s} \mathbb{E}[g(X_r^x)|\mathcal{F}_t]dr$$
$$= \int_t^{t+s} Q_{r-t}g(X_t^x)dr = \int_0^s Q_r g(X_t^x)dr.$$

It follows from the last two displays that

$$\mathbb{E}[h(X_{t+s}^x) - \int_0^{t+s} g(X_r^x)dr|\mathcal{F}_t] = h(X_t^x) - \int_0^t g(X_r^x)dr$$

giving property (ii).

Proof of Theorem 6.14 (cont)

(ii)$\Rightarrow$ (i) Suppose that (ii) holds. Then, for every $t \geq 0$,

$$\mathbb{E}\left[h(X_t^x) - \int_0^t g(X_r^x)dr\right] = h(x)$$

and on the other hand, from the definition of a Markov process,

$$\mathbb{E}\left[h(X_t^x) - \int_0^t g(X_r^x)dr\right] = Q_t h(x) - \int_0^t Q_r g(x)dr.$$

Consequently, as $t \to 0$,

$$\frac{Q_t h - h}{t} = \frac{1}{t}\int_0^t Q_r g dr \to g$$

in $C_0(E)$, by property (ii) of the definition of a Feller semigroup. We conclude that $h \in D(L)$ and $Lh = g$.

Example

In the case of d-dimensional Brownian motion, Ito's formula shows that,

$$h(X_t) - \frac{1}{2}\int_0^t \Delta h(X_s)ds$$

is a continuous local martingale. This continuous local martingale is a martingale if we furthermore assume that h and Δh are in $C_0(\mathbb{R}^d)$. It then follows from Theorem 6.14 that $h \in D(L)$ and $Lh = \frac{1}{2}\Delta h$. Recall that we already obtained this result by a direct computation of L when $d = 1$ (in fact in a more precise form since here we can only assert that $\{h \in C^2(\mathbb{R}^d) : h, \Delta h \in C_0(\mathbb{R}^d)\} \subset D(L)$ whereas equality holds if $d = 1$).

Outline

Our aim in this section is to show that one can construct Feller processes in such a way that they have cadlag sample paths. We consider a Feller semigroup $(Q_t)_{t\geq 0}$ on a locally compact separable metric space.

Theorem 6.15

Let $(X_t)_{t\geq 0}$ be a Markov process with semigroup $(Q_t)_{t\geq 0}$, with respect to the filtration $(\mathcal{F}_t)_{t\in[0,\infty]}$. Set $\widetilde{\mathcal{F}}_\infty = \mathcal{F}_\infty$, and, for every $t \geq 0$,

$$\widetilde{\mathcal{F}}_t = \mathcal{F}_{t+} \vee \sigma(\mathcal{N})$$

where $\mathcal{N}$ is the family of all $\mathcal{F}_\infty$-measurable sets of zero probability. Then, the process $(X_t)_{t\geq 0}$ has a cadlag modification $(\widetilde{X}_t)_{t\geq 0}$, which is adapted to the filtration $(\widetilde{\mathcal{F}}_t)$. Moreover, $(\widetilde{X}_t)_{t\geq 0}$ is a Markov process with semigroup $(Q_t)_{t\geq 0}$, with respect to the filtration $(\widetilde{\mathcal{F}}_t)_{t\in[0,\infty]}$.

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where $\mathcal{N}$ is the family of all $\mathcal{F}_\infty$-measurable sets of zero probability. Then, the process $(X_t)_{t\geq 0}$ has a cadlag modification $(\widetilde{X}_t)_{t\geq 0}$, which is adapted to the filtration $(\widetilde{\mathcal{F}}_t)$. Moreover, $(\widetilde{X}_t)_{t\geq 0}$ is a Markov process with semigroup $(Q_t)_{t\geq 0}$, with respect to the filtration $(\widetilde{\mathcal{F}}_t)_{t\in[0,\infty]}$.

Remark

The filtration $(\widetilde{\mathcal{F}}_t)$ is right-continuous because the filtration $(\mathcal{F}_{t+})$ is, and the right-continuity property is preserved when adding the class of negligible sets $\mathcal{N}$.

Proof of Theorem 6.15

Let $E_\Delta = E \cup \{\Delta\}$ be the one-point compactification of E, which is obtained by adding the point at infinity Δ to E (and by definition the neighborhoods of Δ are the complements of compact subsets of E). We agree that every function $f \in C_0(E)$ is extended to a continuous function on E_Δ by setting $f(\Delta) = 0$.

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Proof of Theorem 6.15 (cont)

Let $C_0^+(E)$ be the set of all non-negative functions in $C_0(E)$. We can find a sequence $(f_n)_{n\geq 0}$ in $C_0^+(E)$ which separates the points of E_Δ, in the sense that, for every $x, y \in E_\Delta$, there is an integer n such that $f_n(x) \neq f_n(y)$. Then

$$\mathcal{H} = \{R_p f_n : p \geq 1, n \geq 0\}$$

is also a countable subset of $C_0^+(E)$ which separates the points of E_Δ (use the fact that $\|pR_p f_n - f_n\| \to 0$ as $p \to \infty$).

If $h \in \mathcal{H}$, Lemma 6.6 shows that there exists an integer $p \geq 1$ such that $e^{-pt} h(X_t)$ is a supermartingale. Let D be a countable dense subset of $\mathbb{R}_+$. Then Theorem 3.17 (i) shows that the limits

$$\lim_{D \ni s \downarrow\downarrow t} h(X_s), \qquad \lim_{D \ni s \uparrow\uparrow t} h(X_s)$$

exist simultaneously for every $t \in \mathbb{R}_+$ (the second one only for $t > 0$) outside an $\mathcal{F}_\infty$-measurable event N_h of zero probability.

Proof of Theorem 6.15 (cont)

Indeed, as in the proof of Theorem 3.17, we may define the complementary event N_h^c as the set of all $\omega \in \Omega$ for which the function $D \ni s \mapsto e^{-ps}h(X_s)$ makes a finite number of upcrossings along any interval $[a, b]$ $(a, b \in \mathbb{Q}, a < b)$ on every finite time interval. We then set

$$N = \cup_{h \in \mathcal{H}} N_h,$$

then we still have $N \in \mathcal{N}$. Then if $\omega \notin N$, the limits

$$\lim_{D \ni s \downarrow\downarrow t} X_s, \qquad \lim_{D \ni s \uparrow\uparrow t} X_s$$

exist simultaneously for every $t \in \mathbb{R}_+$ (the second one only for $t > 0$) in E_Δ. In fact, if we assume that $X_s(\omega)$ has two distinct accumulation points in E_Δ as $D \ni s \downarrow\downarrow t$, we get a contradiction by considering a function $h \in \mathcal{H}$ that separates these two points.

Proof of Theorem 6.15 (cont)

We can then set, for every $\omega \in \Omega \setminus N$ and every $t \geq 0$,

$$\widetilde{X}_t(\omega) = \lim_{D \ni s \downarrow\downarrow t} X_s(\omega).$$

If $\omega \in N$, we set $\widetilde{X}_t(\omega) = x_0$ for every $t \geq 0$, where x_0 is a fixed point in E. Then, for every $t \geq 0$, $\widetilde{X}_t$ is an $\widetilde{\mathcal{F}}_t$-measurable random variable with values in E_Δ. Furthermore, for every $\omega \in \Omega$, $t \mapsto \widetilde{X}_t(\omega)$, viewed as a mapping with values in E_Δ, is cadlag by Lemma 3.16 (this lemma shows that the functions $t \mapsto h(\widetilde{X}_t(\omega))$ is cadlag, and this suffices since $\mathcal{H}$ separates points of E).

Proof of Theorem 6.15 (cont)

Let us now show that $\mathbb{P}(X_t = \widetilde{X}_t) = 1$ for every fixed $t \geq 0$. Let $f, g \in C_0(E)$ and let (t_n) be a sequence in D that decreases (strictly) to t. Then,

$$\begin{aligned}\mathbb{E}[f(X_t)g(\widetilde{X}_t)] &= \lim_{n\to\infty} \mathbb{E}[f(X_t)g(X_{t_n})] \\ &= \lim_{n\to\infty} \mathbb{E}[f(X_t)Q_{t_n - t}g(X_t)] = \mathbb{E}[f(X_t)g(X_t)]\end{aligned}$$

since $Q_{t_n - t}g \to g$ by the definition of a Feller semigroup. The preceding equality entails that the two pairs $(X_t, \widetilde{X}_t)$ and (X_t, X_t) have the same distribution and thus $\mathbb{P}(X_t = \widetilde{X}_t) = 1$.

Proof of Theorem 6.15 (cont)

Let us now show that $(\widetilde{X}_t)_{t\geq 0}$ is a Markov process with semigroup $(Q_t)_{t\geq 0}$ with respect to the filtration $(\widetilde{\mathcal{F}}_t)$. It suffices to prove that, for every $s \geq 0$, $t > 0$, $A \in \widetilde{\mathcal{F}}_s$ and $f \in C_0(E)$,

$$\mathbb{E}[1_A f(\widetilde{X}_{s+t})] = \mathbb{E}[1_A Q_t f(\widetilde{X}_s)].$$

Since $\widetilde{X}_s = X_s$ a.s. and $\widetilde{X}_{s+t} = X_{s+t}$ a.s, this is equivalent to proving that

$$\mathbb{E}[1_A f(X_{s+t})] = \mathbb{E}[1_A Q_t f(X_s)].$$

Because A is equal a.s. to an element of $\mathcal{F}_{s+}$, we may assume that $A \in \mathcal{F}_{s+}$. Let (s_n) be a sequence in D that decreases to s, so that $A \in \mathcal{F}_{s_n}$ for every n. Then, as soon as $s_n \leq s + t$, we have

$$\mathbb{E}[1_A f(X_{s+t})] = \mathbb{E}[1_A \mathbb{E}[f(X_{s+t})|\mathcal{F}_{s_n}]] = \mathbb{E}[1_A Q_{s+t-s_n} f(X_{s_n})].$$

Proof of Theorem 6.15 (cont)

But $Q_{s+t-s_n}f$ converges (uniformly) to $Q_t f$ by properties of Feller semigroups, and since $X_{s_n} = \widetilde{X}_{s_n}$ a.s. we also know that X_{s_n} converges a.s. to $\widetilde{X}_s = X_s$. We thus obtain the desired result by letting n tend to ∞.

It remains to show that the sample paths $t \mapsto \widetilde{X}_t(\omega)$ are cadlag as E-valued mappings, and not only as E_Δ-valued mappings (we already know that, for every fixed $t \geq 0$, $X_t(\omega) = \widetilde{X}_t(\omega)$ a.s. is in E with probability one, but this does not imply that the sample paths, and their left-limits, remain in E). Fix a function $g \in C_0^+(E)$ such that $g(x) > 0$ for every $x \in E$. The function $h = R_1 g$ then satisfies the same property. Set, for every $t \geq 0$,

$$Y_t = e^{-t} h(\widetilde{X}_t).$$

Then Lemma 6.6 shows that $(Y_t)_{t \geq 0}$ is a non-negative supermartingale with respect to the filtration $(\widetilde{\mathcal{F}}_t)$. Additionally, we know that the sample paths of $(Y_t)_{t \geq 0}$ are cadlag (recall that $h(\Delta) = 0$ by convention).

Proof of Theorem 6.15 (cont)

For every integer $n \geq 1$, set

$$T_{(n)} = \inf\{t \geq 0 : Y_t < \frac{1}{n}\}.$$

Then $T_{(n)}$ is a stopping time of the filtration $(\widetilde{\mathcal{F}}_t)$. Consequently,

$$T = \lim_{n\to\infty} \uparrow T_{(n)}$$

is a stopping time. The desired result will follow if we can verify that $\mathbb{P}(T < \infty) = 0$. Indeed, it is clear that, for every $t \in [0, T_{(n)})$, $\widetilde{X}_t \in E$ and $\widetilde{X}_{t-} \in E$ and we may redefine $\widetilde{X}_t(\omega) = x_0$ (for every $t \geq 0$) for all ω belonging to $\{T < \infty\} \in \mathcal{N}$.

Proof of Theorem 6.15 (cont)

To show $\mathbb{P}(T < \infty) = 0$, we apply Theorem 3.25 and the subsequent remark to $Z = Y$ and $U = T_{(n)}$ and $V = T + q$, where $q > 0$ s a rational number. We get

$$\mathbb{E}[Y_{T+q} 1_{\{T<\infty\}}] \leq \mathbb{E}[Y_{T_{(n)}} 1_{\{T_{(n)}<\infty\}}] \leq \frac{1}{n}.$$

By letting n tend to ∞, we get

$$\mathbb{E}[Y_{T+q} 1_{\{T<\infty\}}] = 0,$$

hence $Y_{T+q} = 0$ a.s. on $\{T < \infty\}$. By the right-continuity of sample paths of Y, we conclude that $Y_t = 0$, for every $t \in [T, \infty)$, a.s. on $\{T < \infty\}$. But we also know that, for every integer $k \geq 0$, $Y_k = e^{-k} h(\widetilde{X}_k) > 0$ a.s. since $\widetilde{X}_k \in E$. This suffices to get $\mathbb{P}(T < \infty) = 0$.

Remark

The proof above applies with minor modifications to the different setting where we are given the process $(X_t)_{t\geq 0}$ together with a collection $(\mathbb{P}_x)_{x\in E}$ of probability measures such that, under $\mathbb{P}_x$, $(X_t)_{t\geq 0}$ is a Markov process with semigroup $(Q_t)_{t\geq 0}$, with respect to a filtration $(\mathcal{F}_t)_{t\in[0,\infty]}$. and $\mathbb{P}(X_0 = x) = 1$. In this setting, we can define the filtration $(\widetilde{\mathcal{F}}_t)_{t\in[0,\infty]}$ by

$$\widetilde{\mathcal{F}}_t = \mathcal{F}_{t+} \vee \sigma(\mathcal{N}')$$

where $\mathcal{N}'$ is the family of all the $\mathcal{F}_\infty$-measurable sets that have zero $\mathbb{P}_x$ probability for every $x \in E$. By the same arguments as in the preceding proof, we can then construct an $(\widetilde{\mathcal{F}}_t)$-adapted process $(\widetilde{X}_t)$ with cadlag sample paths, such that, for every $x \in E$,

$$\mathbb{P}_x(\widetilde{X}_t = X_t) = 1, \quad \forall t \geq 0,$$

and $(\widetilde{X}_t)_{t\geq 0}$ is under $\mathbb{P}_x$ a Markov process with semigroup (Q_t), with respect to the filtration $(\widetilde{\mathcal{F}}_t)_{t\in[0,\infty]}$, such that $\mathbb{P}(\widetilde{X}_0 = x) = 1$.