Math 562 Fall 2020

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October 30, 2020

Outline

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HW5 is due Friday Nov. 6 at noon. Please submit your homework via the course Moodle page. Make sure that your HW is uploaded successfully.

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Definition 6.5

Let $\lambda > 0$. The λ-resolvent of the transition semigroup $(Q_t)_{t\geq 0}$ is the linear operator $R_\lambda : B(E) \to B(E)$ defined by

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t} Q_t f(x) dt$$

for $f \in B(E)$ and $x \in E$.

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Properties of the resolvent

(i) $\|R_\lambda f\| \le \frac{1}{\lambda}\|f\|$.

(ii) If $0 \le f \le 1$, then $0 \le \lambda R_\lambda f \le 1$.

(iii) If $\lambda, \mu > 0$, we have $R_\lambda - R_\mu + (\lambda - \mu) R_\lambda R_\mu = 0$ (resolvent equation).

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(i) and (ii) are easy. So we only prove (iii). We may assume $\lambda \neq \mu$. Then

$$R_\lambda R_\mu(x) = \int_0^\infty e^{-\lambda s} Q_s \left(\int_0^\infty e^{-\mu t} Q_t f \right)(x) ds$$
$$= \int_0^\infty e^{-\lambda s} \int_0^\infty e^{-\mu t} Q_{t+s} f(x) dt ds$$
$$= \int_0^\infty e^{-(\lambda-\mu)s} \int_0^\infty e^{-\mu(t+s)} Q_{t+s} f(x) dt ds$$
$$= \int_0^\infty e^{-(\lambda-\mu)s} \int_s^\infty e^{-\mu r} Q_r f(x) dr ds$$
$$= \int_0^\infty Q_r f(x) e^{-\mu r} \int_0^r e^{-(\lambda-\mu)s} ds dr$$
$$= \int_0^\infty Q_r f(x) \left(\frac{e^{-\mu r} - e^{-\lambda r}}{\lambda - \mu} \right) dr$$

giving the desired result.

Example

In the case of 1-dim Brownian motion,

$$R_\lambda f(x) = \int_{-\infty}^{\infty} r_\lambda(y-x) f(y) dy$$

where

$$r_\lambda(y-x) = \int_0^\infty (2\pi t)^{-1/2} \exp\left(-\frac{|y-x|^2}{2t} - \lambda t\right) dt = \frac{1}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}|y-x|},$$

where the last equality is elementary. You could also justify the last equality using the formula

$$\mathbb{E}[e^{-\lambda T_a}] = e^{-a\sqrt{2\lambda}}$$

for the Laplace transform of the hitting time T_a of $a > 0$ by a 1-dim Brownian motion.

A key motivation of the introduction of the resolvent is the fact that it allows one to construct certain supermartingales associated with a Markov process.

Lemma 6.6

Let X be a Markov process with semigroup $(Q_t)_{t\geq 0}$ with respect to the filtration $(\mathcal{F}_t)$. Let $h \in B(E)$ be non-negative and let $\lambda > 0$. Then the process

$$e^{-\lambda t} R_\lambda h(X_t)$$

is an $(\mathcal{F}_t)$-supermartingale.

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Proof of Lemma 6.6

The random variables $e^{-\lambda t} R_\lambda h(X_t)$ are bounded and thus in L^1. Then, for every $s \geq 0$,

$$Q_s R_\lambda h = \int_0^\infty e^{-\lambda t} Q_{t+s} h dt$$

and it follows that

$$e^{-\lambda s} Q_s R_\lambda h = \int_0^\infty e^{-\lambda(t+s)} Q_{t+s} h dt = \int_s^\infty e^{-\lambda t} Q_t h dt \leq R_\lambda h.$$

Hence, for every $s, t \geq 0$,

$$\mathbb{E}[e^{-\lambda(t+s)} R_\lambda h(X_{t+s}) | \mathcal{F}_t] = e^{-\lambda(t+s)} Q_s R_\lambda h(X_t) \leq e^{-\lambda t} R_\lambda h(X_t)$$

giving the desired supermartingale property.

Outline

From now on, we assume that E is a metrizable locally compact topological space. We equip E with its Borel σ-field $\mathcal{E}$. There exists an increasing sequence $(K_n)_{n\geq 1}$ of compact subsets of E such that $E = \cup_n K_n$. Any compact set of E is contained in K_n for some n.

A function $f : E \to \mathbb{R}$ is said to tend to 0 at infinity if, for every $\epsilon > 0$, there exists a compact subset K of E such that $|f(x)| \leq \epsilon$ for all $x \in K^c$. This is equivalent to requiring that

$$\sup_{x \in K_n^c} |f(x)| \to 0, \quad \text{as } n \to \infty.$$

We use $C_0(E)$ to denote the set of all continuous real-valued functions on E that tend to 0 at infinity. The space $C_0(E)$ is a Banach space for the supremum norm

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Definition 6.7

Let $(Q_t)_{t\geq 0}$ be a transition semigroup on E. We say that $(Q_t)_{t\geq 0}$ is a Feller semigroup if

(i) $\forall f \in C_0(E)$, $Q_t f \in C_0(E)$;

(ii) $\forall f \in C_0(E)$, $\|Q_t f - f\| \to 0$ as $t \to 0$.

A Markov process with values in E is a Feller process if its semigroup if Feller.

By using the fact that the dual space of $C_0(E)$ is the space of finite Borel measures on E and the Hahn-Banach theorem, one can prove that condition (ii) can be replaced by the seemingly weaker property

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Condition (ii) implies that, for every $s \geq 0$,

$$\lim_{t \downarrow 0} \|Q_{s+t}f - Q_s f\| = \lim_{t \downarrow 0} \|Q_s(Q_t f - f)\| = 0.$$

since Q_s is a contraction on $C_0(E)$. Note that the convergence is uniform when s varies over $\mathbb{R}_+$, which ensures that the mapping $t \mapsto Q_t f$ is uniformly continuous from $\mathbb{R}_+$ to $C_0(E)$, for any fixed $f \in C_0(E)$.

In what follows, we fix a Feller semigroup $(Q_t)_{t \geq 0}$ on E. Using property (i) of the definition and the dominated convergence theorem, one easily verifies that $R_\lambda f \in C_0(E)$ for every $f \in C_0(E)$ and $\lambda > 0$.

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Proposition 6.8

Let $\lambda > 0$ and set $\mathcal{R} = \{R_\lambda f : f \in C_0(E)\}$. Then $\mathcal{R}$ does not depend on the choice $\lambda > 0$. Furthermore, $\mathcal{R}$ is a dense subspace of $C_0(E)$.

Proof of Proposition 6.8

If $\lambda \neq \mu$, the resolvent equation gives

$$R_\lambda f = R_\mu(f + (\mu - \lambda)R_\lambda f).$$

Hence any function of the form $R_\lambda f$ with $f \in C_0(E)$ is also of the form $R_\mu g$ for some $g \in C_0(E)$. This gives the first assertion.
Clearly, $\mathcal{R}$ is a linear subspace of $C_0(E)$. To see that it is dense, we simply note that, for every $f \in C_0(E)$, as $\lambda \to \infty$,

$$\lambda R_\lambda f = \lambda \int_0^\infty e^{-\lambda t} Q_t f dt = \int_0^\infty e^{-t} Q_{t/\lambda} f dt \to f, \quad \text{in } C_0(E)$$

by property (ii) of the definition of a Feller semigroup and dominated convergence.

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Definition 6.9

We define

$$D(L) = \{f \in C_0(E) : \frac{Q_t f - f}{t} \text{ converges in } C_0(E) \text{ as } t \downarrow 0\}$$

and, for every $f \in D(L)$,

$$Lf = \lim_{t \downarrow 0} \frac{Q_t f - f}{t}.$$

Then $D(L)$ is a linear subspace of $C_0(E)$ and $L : D(L) \to C_0(E)$ is a linear operator called the generator of the semigroup $(Q_t)_{t \geq 0}$. The subspace $D(L)$ is called the domain of L.

Proposition 6.10

Let $f \in D(L)$ and $s > 0$. Then $Q_s f \in D(L)$ and $L(Q_s f) = Q_s(Lf)$.

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Proof of Proposition 6.10

Note that

$$\frac{Q_t(Q_s f) - Q_s f}{t} = Q_S\left(\frac{Q_t f - f}{t}\right).$$

Using the fact that Q_s is a contraction of $C_0(E)$, we get that $t^{-1}(Q_t(Q_s f) - Q_s f)$ converges to $Q_s(Lf)$, which gives the desired result.

Proposition 6.11

If $f \in D(L)$, we have, for every $t \geq 0$,

$$Q_t f = f + \int_0^t Q_s(Lf)ds = f + \int_0^t L(Q_s f)ds.$$

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Proof of Proposition 6.11

Let $f \in D(L)$. For every $t \geq 0$,

$$\epsilon^{-1}(Q_{t+\epsilon}f - Q_t f) = Q_t(\epsilon^{-1}(Q_\epsilon f - f)) \to Q_t(Lf)$$

as $\epsilon \downarrow 0$. Moreover, the preceding convergence is uniform when t varies over $\mathbb{R}_+$. This implies that, for every $x \in E$, the function $t \mapsto Q_t f(x)$ is differentiable on $\mathbb{R}_+$ and its derivative is $Q_t(Lf)(x)$, which is a continuous function of t. The formula of the proposition follows, also using the preceding proposition.

Proposition 6.12

Let $\lambda > 0$.

(i) For every $g \in C_0(E)$, $R_\lambda g \in D(L)$ and $(\lambda - L)R_\lambda g = g$.

(ii) If $f \in D(L)$, $R_\lambda(\lambda - L)f = f$.

Consequently, $D(L) = \mathcal{R}$ and the operators $R_\lambda : C_0(E) \to \mathcal{R}$ and $\lambda - L : D(L) \to C_0(E)$ are the inverse of each other.

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Proof of Proposition 6.12

(i) If $g \in C_0(E)$, we have for every $\epsilon > 0$,

$$\epsilon^{-1}(Q_\epsilon R_\lambda g - R_\lambda g) = \epsilon^{-1}\left(\int_0^\infty e^{-\lambda t} Q_{\epsilon+t} g dt - \int_0^\infty e^{-\lambda t} Q_t g dt\right)$$

$$= \epsilon^{-1}\left((1 - e^{-\lambda\epsilon})\int_0^\infty e^{-\lambda t} Q_{\epsilon+t} g dt - \int_0^\epsilon e^{-\lambda t} Q_{\epsilon+t} g dt\right)$$

$$\to \lambda R_\lambda g - g$$

as $\epsilon \downarrow 0$. using property (ii) of the definition of a Feller semigroup (and the fact that this property implies the continuity of the mapping $t \mapsto Q_t g$ from $\mathbb{R}_+$ to $C_0(E)$). The preceding calculation shows that $R_\lambda g \in D(L)$ and $(\lambda - L)R_\lambda g = g$.

Proof of Proposition 6.12 (cont)

(ii) Let $F \in D(L)$. By Proposition 6.11, $Q_t f = f + \int_0^t Q_s(Lf)ds$, hence

$$\int_0^\infty e^{-\lambda t} Q_t f(x)dt = \frac{f(x)}{\lambda} + \int_0^\infty e^{-\lambda t} \left(\int_0^t Q_s(Lf)(x)ds \right) dt$$
$$= \frac{f(x)}{\lambda} + \int_0^\infty \frac{e^{-\lambda s}}{\lambda} Q_s(Lf)(x)ds.$$

We have thus obtained the equality

$$\lambda R_\lambda f = f + R_\lambda Lf$$

giving the result in (ii).

Corollary 6.13

The semigroup $(Q_t)_{t\geq 0}$ is determined by the generator L (including also the domain $D(L)$).

Proof of Corollary 6.13

Let f be a non-negative function in $C_0(E)$. Then $R_\lambda f$ is the unique element of $D(L)$ such that $(\lambda - L)R_\lambda f = f$. On the other hand, knowing $R_\lambda f(x) = \int_0^\infty e^{-\lambda t} Q_t f(x) dt$ for every $\lambda > 0$ determines the continuous function $t \mapsto Q_t f(x)$. To complete the argument, note that Q_t is characterized by the values of $Q_t f$ for every non-negative function f in $C_0(E)$

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