# Math 562 Fall 2020

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# Outline

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HW4 has been graded.

I will set up HW5 later today. The due date is Nov. 6 at noon.

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**Theorem 5.22 (Girsanov's theorem)**

Assume that the probability measures $\mathbb{P}$ and $\mathbb{Q}$ are mutually absolutely continuous on $\mathcal{F}_\infty$. Let $(D_t)_{t\geq 0}$ be the martingale with cadlag sample paths such that, for every $t \geq 0$,

$$D_t = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t}.$$

Assume that $D$ has continuous sample paths, and let $L$ be the unique continuous local martingale such that $D_t = \mathcal{E}(L)_t$. Then, if $M$ is a continuous local martingale under $\mathbb{P}$, the process

$$\widetilde{M} = M - \langle M, L\rangle$$

is a continuous local martingale under $\mathbb{Q}$.

In most applications of Girsanov's theorem, one constructs the probability measure $\mathbb{Q}$ in the following way. Start from a continuous local martingale $L$ such that $L_0 = 0$ and $\langle L, L\rangle_\infty < \infty$ a.s. The latter condition implies that the limit $L_\infty := \lim_{t\to\infty} L_t$ exists a.s. Then $\mathcal{E}(L)_t$ is a nonnegative continuous local martingale hence a supermartingale, which converges a.s. to

$$\mathcal{E}(L)_\infty = \exp(L_\infty - \frac{1}{2}\langle L, L\rangle_\infty),$$

and $\mathbb{E}[\mathcal{E}(L)_\infty] \leq 1$ by Fatou's lemma. If the property

$$\mathbb{E}[\mathcal{E}(L)_\infty] = 1 \tag{1}$$

holds, then $\mathcal{E}(L)$ is a uniformly integrable martingale (by Fatou's lemma again, one has $\mathcal{E}(L)_t \geq \mathbb{E}[\mathcal{E}(L)_\infty|\mathcal{F}_t]$, but (1) implies that $\mathbb{E}[\mathcal{E}(L)_\infty] = \mathbb{E}[\mathcal{E}(L)_0] = \mathbb{E}[\mathcal{E}(L)_t]$ for every $t \geq 0$). If we let $\mathbb{Q}$ be the probability measure with density $\mathcal{E}(L)_\infty$, with respect to $\mathbb{P}$, we are in the setting of Theorem 5.22, with $D_t = \mathcal{E}(L)_t$. It is therefore very important to give conditions that ensure that (1) holds.

**Theorem 5.23**

Let $L$ be a continuous local martingale such that $L_0 = 0$. Consider the following properties:

**(i)** $\mathbb{E}[\exp(\frac{1}{2}\langle L, L\rangle_\infty)] < \infty$ (Novikov's condition);

**(ii)** $L$ is a uniformly integrable martingale, and $\mathbb{E}[\exp(\frac{1}{2}L_\infty)] < \infty$ (Kazamaki's condition);

**(iii)** $\mathcal{E}(L)$ is a uniformly integrable martingale.

Then (i)$\Rightarrow$(ii) $\Rightarrow$(iii).

## Proof of Theorem 5.23

(i)⇒(ii): Property (i) implies that $\mathbb{E}[\langle L, L\rangle_\infty] < \infty$ hence also that L is a continuous martingale bounded in $L^2$. Then,

$$\exp(\frac{1}{2}L_\infty) = (\mathcal{E}(L)_\infty)^{1/2}(\exp(\frac{1}{2}\langle L, L\rangle_\infty))^{1/2}$$

so that, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\mathbb{E}[\exp(\frac{1}{2}L_\infty)] &\leq (\mathbb{E}[\mathcal{E}(L)_\infty])^{1/2}(\mathbb{E}[\exp(\frac{1}{2}\langle L, L\rangle_\infty)])^{1/2}\\ &\leq (\mathbb{E}[\exp(\frac{1}{2}\langle L, L\rangle_\infty)])^{1/2} < \infty.\end{aligned}$$

## Proof of Theorem 5.23 (cont)

(ii)$\Rightarrow$(iii): Since $L$ is a uniformly integrable martingale, Theorem 3.22 shows that, for any stopping time $T$, we have $L_T = \mathbb{E}[L_\infty | \mathcal{F}_T]$. Jensen's inequality then gives

$$\exp(\frac{1}{2}L_T) \leq \mathbb{E}[\exp(\frac{1}{2}L_\infty)|\mathcal{F}_T].$$

By assumption, $\mathbb{E}[\exp(\frac{1}{2}L_\infty)] < \infty$ which implies that the collection of all variables of the form $\mathbb{E}[\exp(\frac{1}{2}L_\infty)|\mathcal{F}_T]$, for any stopping time $T$, is uniformly integrable. The preceding bound then shows that the collection of all variables $\exp(\frac{1}{2}L_T)$, for any stopping time $T$, is also uniformly integrable.

For $0 < a < 1$, define $Z_t^{(a)} = \exp(\frac{aL_t}{1+a})$. Then, one easily verifies that

$$\mathcal{E}(aL)_t = (\mathcal{E}(L)_t)^{a^2}(Z_t^{(a)})^{1-a^2}.$$

## Proof of Theorem 5.23 (cont)

If $\Gamma \in \mathcal{F}$ and $T$ is a stopping time, Hölder's inequality gives

$$\mathbb{E}[1_\Gamma \mathcal{E}(aL)_T] \le (\mathbb{E}[\mathcal{E}(L)_T])^{a^2} (\mathbb{E}[1_\Gamma Z_T^{(a)}])^{1-a^2}$$
$$\le (\mathbb{E}[1_\Gamma Z_T^{(a)}])^{1-a^2} \le (\mathbb{E}[1_\Gamma \exp(\frac{1}{2} L_T)])^{2a(1-a)}.$$

In the second inequality, we used the property $\mathbb{E}[\mathcal{E}(L)_T] \le 1$, which holds by Proposition 3.25 because $\mathcal{E}(L)$ is a nonnegative supermartingale and $\mathcal{E}(L)_0 = 1$. In the third inequality, we use Jensen's inequality, noting that $\frac{1+a}{2a} > 1$ since the collection of all variables of the form $\exp(\frac{1}{2} L_T)$, for any stopping time $T$, is uniformly integrable, the preceding display shows that so is the collection of all variables $\mathcal{E}(aL)_T$, for any stopping time $T$.

## Proof of Theorem 5.23 (cont)

By the definition of a continuous local martingale, there is an increasing sequence $T_n \uparrow \infty$ of stopping times, such that, for every $n$, $\mathcal{E}(aL)_{t\wedge T_n}$ is a martingale. If $0 \le s \le t$, we can use uniform integrability to pass to the limit $n \to \infty$ in the equality $\mathbb{E}[\mathcal{E}(aL)_{t\wedge T_n}|\mathcal{F}_s] = \mathcal{E}(aL)_{s\wedge T_n}$ and we get that $\mathcal{E}(aL)$ is a uniformly integrable martingale. It follows that

$$\begin{aligned} 1 = \mathbb{E}[\mathcal{E}(aL)_\infty] &\le (\mathbb{E}[\mathcal{E}(L)_\infty])^{a^2}(\mathbb{E}[Z_\infty^{(a)}])^{1-a^2} \\ &\le (\mathbb{E}[\mathcal{E}(L)_\infty])^{a^2}(\mathbb{E}[\exp(\frac{1}{2}L)_\infty])^{2a(1-a)}, \end{aligned}$$

using again Jensen's inequality as above. When $a \uparrow 1$, this gives $\mathbb{E}[\mathcal{E}(L)_\infty] \ge 1$ hence $\mathbb{E}[\mathcal{E}(L)_\infty] = 1$.

# Outline

In this section, we describe a few applications of Girsanov's theorem, which illustrate the strength of the previous results.

## Constructing solutions of stochastic differential equations

Let $b$ be a bounded measurable function on $\mathbb{R}_+ \times \mathbb{R}$. We assume that there exists a function $g \in L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dt)$ such that $|b(t,x)| \leq g(t)$ for every $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. This holds in particular if there exists an $A > 0$ such that $|b|$ is bounded on $[0, A] \times \mathbb{R}$ and vanishes on $(A, \infty) \times \mathbb{R}$.

Let $B$ be an $(\mathcal{F}_t)$-Brownian motion. Consider the continuous local martingale

$$L_t = \int_0^t b(s, B_s) dB_s$$

and the associated exponential martingale

$$D_t = \mathcal{E}(L)_t = \exp\left(\int_0^t b(s, B_s) dB_s - \frac{1}{2}\int_0^t b^2(s, B_s) ds\right).$$

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## Constructing solutions of stochastic differential equations

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Our assumption on $b$ ensures that condition (i) of Theorem 5.23 holds, and thus $D$ is a uniformly integrable martingale. We set $\mathbb{Q} = D_\infty \cdot \mathbb{P}$. Girsanov's theorem, and remark (c) following the statement of the theorem, show that the process

$$\beta_t := B_t - \int_0^t b(s, B_s) ds$$

is an $(\mathcal{F}_t)$-Brownian motion under $\mathbb{Q}$.

We can restate the latter property by saying that, under the probability measure $\mathbb{Q}$, there exists an $(\mathcal{F}_t)$-Brownian motion $\beta$ such that the process $X = B$ solves the stochastic differential equation

$$dX_t = d\beta_t + b(t, B_t)dt.$$

This equation is of the type that will be considered in Chap. 7 below. Note that we are not making any regularity assumption on the function $b$. It is remarkable that Girsanov's theorem allows one to construct solutions of stochastic differential equations without regularity conditions on the coefficients.

## The Cameron-Martin formula

We now assume that $b(t,x) = g(t)$, where $g \in L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dt)$, and we also set, for every $t \geq 0$,

$$h(t) = \int_0^t g(s)ds.$$

The set $\mathcal{H}$ of all functions h that can be written in this form is called the Cameron-Martin space. If $h \in \mathcal{H}$, we sometimes write $\dot{h} = g$ for the associated function in $L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dt)$ (this is the derivative of $h$ in the sense of distributions).
As a special case of the previous discussion, under the probability measure

$$\mathbb{Q} := D_\infty \cdot \mathbb{P} = \exp\left(\int_0^\infty g(t)dB_t - \frac{1}{2}\int_0^\infty g^2(t)dt\right) \cdot \mathbb{P},$$

the process $\beta_t = B_t - h(t)$ is a Brownian motion. Hence, for every non-negative measurable function $\Phi$ on $C(\mathbb{R}_+, \mathbb{R})$,

$$\mathbb{E}_{\mathbb{P}}[D_\infty \cdot \Phi((B_t)_{t\geq 0})] = \mathbb{E}_{\mathbb{Q}}[\Phi((B_t)_{t\geq 0})]$$
$$= \mathbb{E}_{\mathbb{Q}}[\Phi((\beta_t + h(t))_{t\geq 0})] = \mathbb{E}_{\mathbb{P}}[\Phi((B_t + h(t))_{t\geq 0})].$$

The equality between the two ends of the last display is the Cameron-Martin formula. In the next proposition, we write this formula in the special case of the canonical construction of Brownian motion on the Wiener space.

**Proposition 5.24 (Cameron-Martin formula)**

Let $W(dw)$ be the Wiener measure on $C(\mathbb{R}_+, \mathbb{R})$, and let $h$ be a function in the Cameron-Martin space $\mathcal{H}$. Then, for every non-negative measurable function $\Phi$ on $C(\mathbb{R}_+, \mathbb{R})$,

$$\int W(dw)\Phi(w+h) = \int W(dw) \exp\left(\int_0^\infty \dot{h}(s)dw_s - \frac{1}{2}\int_0^\infty \dot{h}(s)^2 ds\right) \Phi(w).$$

non-negative measurable function $\Phi$ on $C(\mathbb{R}_+, \mathbb{R})$,

$$\mathbb{E}_{\mathbb{P}}[D_\infty \cdot \Phi((B_t)_{t\geq 0})] = \mathbb{E}_{\mathbb{Q}}[\Phi((B_t)_{t\geq 0})]$$
$$= \mathbb{E}_{\mathbb{Q}}[\Phi((\beta_t + h(t))_{t\geq 0})] = \mathbb{E}_{\mathbb{P}}[\Phi((B_t + h(t))_{t\geq 0})].$$

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## Remark

The integral $\int_0^\infty \dot{h}(s)dw_s$ is a stochastic integral with respect to $w(s)$ (which is a Brownian motion under $W(dw)$), but it can also be viewed as a Wiener integral since the function $\dot{h}$ is deterministic. The Cameron-Martin formula can be established by Gaussian calculations that do not involve stochastic integrals or Girsanov's theorem. Still it is instructive to derive this formula as a special case of Girsanov's theorem.

The Cameron-Martin formula gives a "quasi-invariance" property of Wiener measure under the translations by functions of the Cameron-Martin space: The image of the Wiener measure $W(dw)$ under the mapping $w \mapsto w + h$ has a density with respect to $W(dw)$ and this density is the terminal value of the exponential martingale associated with the martingale $\int_0^t \dot{h}(s)dw_s$.

## Law of hitting times for Brownian motion with drift

Let B be a 1-dim Brownian motion with $B_0 = 0$, and for every $a > 0$, let $T_a := \inf\{t \geq 0 : B_t = a\}$. If $c \in \mathbb{R}$, we would like to compute the law of the stopping time

$$U_a := \inf\{t \geq 0 : B_t + ct = a\}.$$

Of course, if $c = 0$, we have $U_a = T_a$, and the desired distribution is given by Corollary 2.22. Girsanov's theorem (or rather the Cameron-Martin formula) will allow us to derive the case where $c$ is arbitrary from the special case $c = 0$.

Fix $t > 0$ and apply the Cameron-Martin formula with

$$\dot{h}(s) = c\mathbf{1}_{\{s \leq t\}}, \quad h(s) = c(s \wedge t).$$

and, for every $w \in C(\mathbb{R}_+, \mathbb{R})$,

$$\Phi(w) = \mathbf{1}_{\{\max_{[0,t]} w(s) > a\}}.$$

It follows that

$$
\begin{aligned}
\mathbb{P}(U_a \le t) &= \mathbb{E}[\Phi(B+h)] \\
&= \mathbb{E}[\Phi(B)\exp\left(\int_0^\infty \dot{h}(s)dB_s - \frac{1}{2}\int_0^\infty \dot{h}(s)^2 ds\right)] \\
&= \mathbb{E}[1_{\{T_a \le t\}} \exp(cB_t - \frac{c^2}{2}t] = \mathbb{E}[1_{\{T_a \le t\}} \exp(cB_{t\wedge T_a} - \frac{c^2}{2}(t\wedge T_a)] \\
&= \mathbb{E}[1_{\{T_a \le t\}} \exp(ca - \frac{c^2}{2}T_a] = \int_0^t ds \frac{a}{\sqrt{2\pi s^3}} e^{-\frac{a^2}{2s}} e^{ca - \frac{c^2}{2}s} \\
&= \int_0^t ds \frac{a}{\sqrt{2\pi s^3}} e^{-\frac{1}{2s}(a-cs)^2}
\end{aligned}
$$

where, in the fourth equality, we used the optional stopping theorem to write

$$
\mathbb{E}[\exp(cB_t - \frac{c^2}{2}t)|\mathcal{F}_{t\wedge T_a}] = \exp(cB_{t\wedge T_a} - \frac{c^2}{2}(t\wedge T_a)
$$

and we also made use of the explicit density of $T_a$ given in Corollary 2.22. This calculation shows that the variable $U_a$ has a density on $\mathbb{R}_+$ given by

$$\psi(s) = \frac{a}{\sqrt{2\pi s^3}} e^{-\frac{1}{2s}(a-cs)^2}.$$

By integrating this density, we get that

$$\mathbb{P}(U_a < \infty) = \begin{cases} 1, & \text{if } c \geq 0; \\ e^{2ca}, & \text{if } c \leq 0, \end{cases}$$

which may also be checked more easily by applying the optional stopping theorem to the continuous martingale $\exp(-2c(B_t + ct))$.