Math 562 Fall 2020

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Outline

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Throughout this section, we assume that the filtration $(\mathcal{F}_t)$ is complete and right-continuous. Our goal is to study how the notions of a martingale and of a semimartingale are affected when the underlying probability measure $\mathbb{P}$ is replaced by another probability measure $\mathbb{Q}$. Most of the time we will assume that $\mathbb{P}$ and $\mathbb{Q}$ are mutually absolutely continuous, and then the fact that the filtration $(\mathcal{F}_t)$ is complete with respect to $\mathbb{P}$ implies that it is complete with respect to $\mathbb{Q}$. When there is a risk of confusion,we will write $\mathbb{E}_{\mathbb{P}}$ for the expectation under the probability measure $\mathbb{P}$, and similarly $\mathbb{E}_{\mathbb{Q}}$ for the expectation under $\mathbb{Q}$. Unless otherwise specified, the notions of a (local)martingale or of a semimartingale refer to the underlying probability measure $\mathbb{P}$ (when we consider these notions under $\mathbb{Q}$ we will say so explicitly). Note that, in contrast with the notion of a martingale, the notion of a finite variation process does not depend on the underlying probability measure.

Proposition 5.20

Assume that $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{F})$, which is absolutely continuous with respect to $\mathbb{P}$ on the σ-field $\mathcal{F}_\infty$. For every $t \in [0, \infty]$, let

$$D_t = \frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t}$$

be the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{F}_t$. The process $(D_t)_{t\geq 0}$ is a uniformly integrable martingale. Consequently $(D_t)_{t\geq 0}$ has a cadlag modification. Keeping the same notation $(D_t)_{t\geq 0}$ for this modification, we have, for every stopping time T,

$$D_T = \frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_T}.$$

Finally, if we assume furthermore that $\mathbb{P}$ and $\mathbb{Q}$ are mutually absolutely continuous on $\mathcal{F}_\infty$, we have

$$\inf_{t\geq 0} D_t > 0, \quad \mathbb{P}_a.s.$$

Proof of Proposition 5.20

If $A \in \mathcal{F}_t$, then

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{Q}}[1_A] = \mathbb{E}_{\mathbb{P}}[1_A D_\infty] = \mathbb{E}_{\mathbb{P}}[1_A \mathbb{E}_{\mathbb{P}}[D_\infty | \mathcal{F}_t]]$$

and, by the uniqueness of the Radon-Nikodym derivative, it follows that

$$D_t = \mathbb{E}_{\mathbb{P}}[D_\infty | \mathcal{F}_t] \quad a.s.$$

Hence D is a uniformly integrable martingale, which is closed by D_∞. Theorem 3.18 (using the fact that $(\mathcal{F}_t)$ is both complete and right-continuous) then allows us to find a cadlag modification of (D_t), which we consider from now on.

Then, if T is a stopping time, the optional stopping theorem gives for every $A \in \mathcal{F}_T$,

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Proof of Proposition 5.20 (cont)

and, since $D_T \in \mathcal{F}_T$, it follows that

$$D_T = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T}.$$

Let us prove the last assertion. For every $\epsilon > 0$, define

$$T_\epsilon = \inf\{t \geq 0 : D_t < \epsilon\}.$$

T_ϵ is a stopping time as the first hitting time of an open set by a cadlad process (using the fact that the filtration is right-continuous). Then, noting that the event $\{T_\epsilon < \infty\}$ is $\mathcal{F}_{T_\epsilon}$-measurable,

$$\mathbb{Q}(T_\epsilon < \infty) = \mathbb{E}_{\mathbb{P}}[1_{\{T_\epsilon < \infty\}} D_{T_\epsilon}] \leq \epsilon$$

since $D_{T_\epsilon} \leq \epsilon$ on $\{T_\epsilon < \infty\}$ by the right-continuity of sample paths.

Proof of Proposition 5.20 (cont)

It immediately follows that

$$\mathbb{Q}\left(\cap_{n=1}^{\infty}\{T_{1/n} < \infty\}\right) = 0.$$

and since $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$ we have also

$$\mathbb{P}\left(\cap_{n=1}^{\infty}\{T_{1/n} < \infty\}\right) = 0.$$

But this exactly means that, $\mathbb{P}$ a.s., there exists an integer $n \geq 1$, such that $T_{1/n} = \infty$, giving the last assertion of the proposition.

Proposition 5.21

Let D be a continuous local martingale taking (strictly) positive values. There exists a unique continuous local martingale L such that

$$D_t = \exp\left(L_t - \frac{1}{2}\langle L, L\rangle_t\right) = \mathcal{E}(L)_t.$$

Moreover, L is given by the formula

$$L_t = \log D_0 + \int_0^t D_s^{-1}\, dD_s.$$

Proof of Proposition 5.21

Uniqueness is an easy consequence of Theorem 4.8. Then, since D takes positive values, we can apply Ito's formula to $\log D_t$ (see the remark before Proposition 5.11), and we get

$$\log D_t = \log D_0 + \int_0^t \frac{dD_s}{D_s} - \frac{1}{2}\int_0^t \frac{d\langle D, D\rangle_s}{D_s^2} = L_t - \frac{1}{2}\langle L, L\rangle_t.$$

where L is as in the statement of the proposition.

We now state the main theorem of this section, which explains the relation between continuous local martingales under $\mathbb{P}$ and continuous local martingales under $\mathbb{Q}$.

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Theorem 5.22 (Girsanov's theorem)

Assume that the probability measures $\mathbb{P}$ and $\mathbb{Q}$ are mutually absolutely continuous on $\mathcal{F}_\infty$. Let $(D_t)_{t\geq 0}$ be the martingale with cadlag sample paths such that, for every $t \geq 0$,

$$D_t = \frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t}.$$

Assume that D has continuous sample paths, and let L be the unique continuous local martingale such that $D_t = \mathcal{E}(L)_t$. Then, if M is a continuous local martingale under $\mathbb{P}$, the process

$$\widetilde{M} = M - \langle M, L \rangle$$

is a continuous local martingale under $\mathbb{Q}$.

Remark

By consequences of the martingale representation theorem explained at the end of the previous section, the continuity assumption for the sample paths of D always holds when $(\mathcal{F}_t)$ is the (completed) canonical filtration of a Brownian motion. In applications of Theorem 5.22, one often starts from the martingale (D_t) to define the probability measure $\mathbb{Q}$, so that the continuity assumption is satisfied by construction.

Proof of Theorem 5.22

The fact that D_t can be written in the form $D_t = \mathcal{E}(L)_t$ follows from Proposition 5.21 (we are assuming that D has continuous sample paths, and we also know from Proposition 5.20 that D takes positive values). Then, let T be a stopping time and let X be an adapted process with continuous sample paths.

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Proof of Theorem 5.22 (cont)

We claim that, if $(XD)^T$ is a martingale under $\mathbb{P}$, then X^T is a martingale under $\mathbb{Q}$. Let us verify the claim. By Proposition 5.20, $\mathbb{E}_{\mathbb{Q}}[|X_{T\wedge t}|] = \mathbb{E}_{\mathbb{P}}[|X_{T\wedge t}|D_{T\wedge t}] < \infty$, and it follows that $X_t^T \in L^1(\mathbb{Q})$. Then, let $A \in \mathcal{F}_s$ and $s < t$. Since $A \cap \{T > s\} \in \mathcal{F}_s$, we have, using the fact that $(XD)^T$ is a martingale under $\mathbb{P}$,

$$\mathbb{E}_{\mathbb{P}}[1_{A\cap\{T>s\}} X_{T\wedge t} D_{T\wedge t}] = \mathbb{E}_{\mathbb{P}}[1_{A\cap\{T>s\}} X_{T\wedge s} D_{T\wedge s}].$$

By Proposition 5.20,

$$D_{T\wedge t} = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_{T\wedge t}}, \quad D_{T\wedge s} = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_{T\wedge s}}$$

and thus, since $A \cap \{T > s\} \in \mathcal{F}_{T\wedge s} \subset \mathcal{F}_{T\wedge t}$, it follows that

$$\mathbb{E}_{\mathbb{Q}}[1_{A\cap\{T>s\}} X_{T\wedge t}] = \mathbb{E}_{\mathbb{Q}}[1_{A\cap\{T>s\}} X_{T\wedge s}].$$

Proof of Theorem 5.22 (cont)

On the other hand, it is immediate that

$$\mathbb{E}_{\mathbb{Q}}[1_{A\cap\{T\leq s\}}X_{T\wedge t}] = \mathbb{E}_{\mathbb{Q}}[1_{A\cap\{T\leq s\}}X_{T\wedge s}].$$

By combining with the previous display, we have $\mathbb{E}_{\mathbb{Q}}[1_A X_{T\wedge t}] = \mathbb{E}_{\mathbb{Q}}[1_A X_{T\wedge s}]$, giving our claim. As a consequence of the claim, we get that, if XD is a continuous local martingale under $\mathbb{P}$, then X is a continuous local martingale under $\mathbb{Q}$.

Next let M be a continuous local martingale under $\mathbb{P}$, and let $\widetilde{M}$ be as in the statement of the theorem. We apply the preceding observation to $X = \widetilde{M}$, noting that, by Ito's formula,

Proof of Theorem 5.22 (cont)

$$\begin{aligned}\widetilde{M}_t D_t &= M_0 D_0 + \int_0^t \widetilde{M}_s dD_s - \int_0^t D_s d\langle M, L\rangle_s + \langle M, D\rangle_t \\ &= M_0 D_0 + \int_0^t \widetilde{M}_s dD_s + \int_0^t D_s dM_s\end{aligned}$$

since $d\langle M, L\rangle_s = D_s^{-1} d\langle M, D\rangle_s$ by Proposition 5.21. We get that $\widetilde{M}D$ is a continuous local martingale under $\mathbb{P}$, and thus $\widetilde{M}$ is a continuous local martingale under $\mathbb{Q}$.

Consequences

(a) A process M which is a continuous local martingale under $\mathbb{P}$ remains a semimartingale under $\mathbb{Q}$, and its canonical decomposition under $\mathbb{Q}$ is $M = \widetilde{M} + \langle M, L\rangle$ (recall that the notion of a finite variation process does not depend on the underlying probability measure). It follows that the class of semimartingales under $\mathbb{P}$ is contained in the class of semimartingales under $\mathbb{Q}$.

In fact these two classes are equal. Indeed, under the assumptions of Theorem 5.22, $\mathbb{P}$ and $\mathbb{Q}$ play symmetric roles, since the Radon-Nikodym derivative of $\mathbb{P}$ with respect to $\mathbb{Q}$ on the σ-field $\mathcal{F}_t$ is D_t^{-1}, which has continuous sample paths if D does.

Furthermore

$$D_t^{-1} = \exp\left(-L_t + \langle L, L\rangle_t - \frac{1}{2}\langle L, L\rangle_t\right) = \exp\left(-\widetilde{L}_t - \frac{1}{2}\langle \widetilde{L}, \widetilde{L}\rangle_t\right) = \mathcal{E}(-\widetilde{L})_t$$

where $\widetilde{L} = L - \langle L, L\rangle$ is a continuous local martingale under $\mathbb{Q}$, and $\langle \widetilde{L}, \widetilde{L}\rangle = \langle L, L\rangle$. So, under the assumptions of Theorem 5.22, the roles of $\mathbb{P}$ and $\mathbb{Q}$ can be interchanged provided D is replaced by D^{-1} and L is replaced by $-\widetilde{L}$.

(b) Let X and Y be two semimartingales (under $\mathbb{P}$ or under $\mathbb{Q}$). The bracket $\langle X, Y \rangle$ is the same under $\mathbb{P}$ and under $\mathbb{Q}$. In fact this bracket is given in both cases by the approximation of Proposition 4.21 (this observation was used implicitly in (a) above).
Similarly, if H is a locally bounded progressive process, the stochastic integral $H \cdot X$ is the same under $\mathbb{P}$ and under $\mathbb{Q}$. To see this it is enough to consider the case when $X = M$ is a continuous local martingale (under $\mathbb{P}$). Write $(H \cdot M)_{\mathbb{P}}$ for the stochastic integral under $\mathbb{P}$ and $(H \cdot M)_{\mathbb{Q}}$ for the one under $\mathbb{Q}$. By linearity,

$$(H \cdot \widetilde{M})_{\mathbb{P}} = (H \cdot M)_{\mathbb{P}} - H \cdot \langle M, L \rangle = (H \cdot M)_{\mathbb{P}} - \langle (H \cdot M)_{\mathbb{P}}, L \rangle$$

and Theorem 5.22 shows that $(H \cdot \widetilde{M})_{\mathbb{P}}$ is a continuous local martingale under $\mathbb{Q}$. Furthermore the bracket of this continuous local martingale with any continuous local martingale N under $\mathbb{Q}$ is equal to $H \cdot \langle M, N \rangle = H \cdot \langle \widetilde{M}, N \rangle$ and it follows from Theorem 5.6 that $(H \cdot \widetilde{M})_{\mathbb{P}} = (H \cdot \widetilde{M})_{\mathbb{Q}}$ and hence also $(H \cdot M)_{\mathbb{P}} = (H \cdot M)_{\mathbb{Q}}$.

With the notation of Theorem 5.22, set $\widetilde{M} = \mathcal{G}_{\mathbb{Q}}^{\mathbb{P}}(M)$. Then $\mathcal{G}_{\mathbb{Q}}^{\mathbb{P}}$ maps the set of all $\mathbb{P}$-continuous local martingales onto the set of all $\mathbb{Q}$-continuous local martingales. One easily verifies, using the remarks in (a) above, that $\mathcal{G}_{\mathbb{P}}^{\mathbb{Q}} \circ \mathcal{G}_{\mathbb{Q}}^{\mathbb{P}} = Id$. Furthermore, the mapping $\mathcal{G}_{\mathbb{Q}}^{\mathbb{P}}$ commutes with the stochastic integral: if H is a locally bounded progressive process, $H \cdot \mathcal{G}_{\mathbb{Q}}^{\mathbb{P}}(M) = \mathcal{G}_{\mathbb{Q}}^{\mathbb{P}}(H \cdot M)$.
(c) Suppose that $M = B$ is an $(\mathcal{F}_t)$-Brownian motion under $\mathbb{P}$, then $\widetilde{B} = B - \langle B, L \rangle$ is a continuous local martingale under $\mathbb{Q}$, with quadratic variation $\langle \widetilde{B}, \widetilde{B} \rangle_t = \langle B, B \rangle_t = t$. By Theorem 5.12, it follows that $\widetilde{B}$ is an $(\mathcal{F}_t)$-Brownian motion under $\mathbb{Q}$.