Outline
5.5 Girsanov’s Theorem
Throughout this section, we assume that the filtration \((\mathcal{F}_t)\) is complete and right-continuous. Our goal is to study how the notions of a martingale and of a semimartingale are affected when the underlying probability measure \(\mathbb{P}\) is replaced by another probability measure \(\mathbb{Q}\). Most of the time we will assume that \(\mathbb{P}\) and \(\mathbb{Q}\) are mutually absolutely continuous, and then the fact that the filtration \((\mathcal{F}_t)\) is complete with respect to \(\mathbb{P}\) implies that it is complete with respect to \(\mathbb{Q}\). When there is a risk of confusion, we will write \(\mathbb{E}_\mathbb{P}\) for the expectation under the probability measure \(\mathbb{P}\), and similarly \(\mathbb{E}_\mathbb{Q}\) for the expectation under \(\mathbb{Q}\). Unless otherwise specified, the notions of a (local)martingale or of a semimartingale refer to the underlying probability measure \(\mathbb{P}\) (when we consider these notions under \(\mathbb{Q}\) we will say so explicitly). Note that, in contrast with the notion of a martingale, the notion of a finite variation process does not depend on the underlying probability measure.
5.5 Girsanov's Theorem

**Proposition 5.20**

Assume that $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{F})$, which is absolutely continuous with respect to $\mathbb{P}$ on the $\sigma$-field $\mathcal{F}_\infty$. For every $t \in [0, \infty]$, let

$$D_t = \frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t}$$

be the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{F}_t$. The process $(D_t)_{t \geq 0}$ is a uniformly integrable martingale. Consequently $(D_t)_{t \geq 0}$ has a cadlag modification. Keeping the same notation $(D_t)_{t \geq 0}$ for this modification, we have, for every stopping time $T$,

$$D_T = \frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_T}.$$

Finally, if we assume furthermore that $\mathbb{P}$ and $\mathbb{Q}$ are mutually absolutely continuous on $\mathcal{F}_\infty$, we have

$$\inf_{t \geq 0} D_t > 0, \quad \mathbb{P}_{a.s.}$$
Proof of Proposition 5.20

If \( A \in \mathcal{F}_t \), then

\[
\mathbb{Q}(A) = \mathbb{E}_\mathbb{Q}[1_A] = \mathbb{E}_\mathbb{P}[1_AD_\infty] = \mathbb{E}_\mathbb{P}[1_A\mathbb{E}_\mathbb{P}[D_\infty|\mathcal{F}_t]]
\]

and, by the uniqueness of the Radon-Nikodym derivative, it follows that

\[
D_t = \mathbb{E}_\mathbb{P}[D_\infty|\mathcal{F}_t] \quad \text{a.s.}
\]

Hence \( D \) is a uniformly integrable martingale, which is closed by \( D_\infty \). Theorem 3.18 (using the fact that \((\mathcal{F}_t)\) is both complete and right-continuous) then allows us to find a cadlag modification of \((D_t)\), which we consider from now on.

Then, if \( T \) is a stopping time, the optional stopping theorem gives for every \( A \in \mathcal{F}_T \),

\[
\mathbb{Q}(A) = \mathbb{E}_\mathbb{Q}[1_A] = \mathbb{E}_\mathbb{P}[1_AD_\infty] = \mathbb{E}_\mathbb{P}[1_A\mathbb{E}_\mathbb{P}[D_\infty|\mathcal{F}_T]] = \mathbb{E}_\mathbb{P}[1_AD_T]
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\]
Proof of Proposition 5.20 (cont)

and, since $D_T \in \mathcal{F}_T$, it follows that

$$D_T = \left. \frac{dQ}{dP} \right|_{\mathcal{F}_T}.$$ 

Let us prove the last assertion. For every $\epsilon > 0$, define

$$T_\epsilon = \inf\{t \geq 0 : D_t < \epsilon\}.$$ 

$T_\epsilon$ is a stopping time as the first hitting time of an open set by a cadlag process (using the fact that the filtration is right-continuous). Then, noting that the event $\{T_\epsilon < \infty\}$ is $\mathcal{F}_{T_\epsilon}$-measurable,

$$Q(T_\epsilon < \infty) = E_P[1_{\{T_\epsilon < \infty\}} D_{T_\epsilon}] \leq \epsilon$$

since $D_{T_\epsilon} \leq \epsilon$ on $\{T_\epsilon < \infty\}$ by the right-continuity of sample paths.
Proof of Proposition 5.20 (cont)

It immediately follows that

\[ \mathbb{Q} \left( \bigcap_{n=1}^{\infty} \{ T_{1/n} < \infty \} \right) = 0. \]

and since \( \mathbb{P} \) is absolutely continuous with respect to \( \mathbb{Q} \) we have also

\[ \mathbb{P} \left( \bigcap_{n=1}^{\infty} \{ T_{1/n} < \infty \} \right) = 0. \]

But this exactly means that, \( \mathbb{P} \) a.s., there exists an integer \( n \geq 1 \), such that \( T_{1/n} = \infty \), giving the last assertion of the proposition.
Proposition 5.21

Let $D$ be a continuous local martingale taking (strictly) positive values. There exists a unique continuous local martingale $L$ such that

$$D_t = \exp \left( L_t - \frac{1}{2} \langle L, L \rangle_t \right) = \mathcal{E}(L)_t.$$ 

Moreover, $L$ is given by the formula

$$L_t = \log D_0 + \int_0^t D_s^{-1} dD_s.$$
Proof of Proposition 5.21

Uniqueness is an easy consequence of Theorem 4.8. Then, since $D$ takes positive values, we can apply Ito’s formula to $\log D_t$ (see the remark before Proposition 5.11), and we get

$$
\log D_t = \log D_0 + \int_0^t \frac{dD_s}{D_s} - \frac{1}{2} \int_0^t \frac{d\langle D, D \rangle_s}{D_s^2} = L_t - \frac{1}{2} \langle L, L \rangle_t.
$$

where $L$ is as in the statement of the proposition.

We now state the main theorem of this section, which explains the relation between continuous local martingales under $\mathbb{P}$ and continuous local martingales under $\mathbb{Q}$. 
Proof of Proposition 5.21

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where \( L \) is as in the statement of the proposition.

We now state the main theorem of this section, which explains the relation between continuous local martingales under \( \mathbb{P} \) and continuous local martingales under \( \mathbb{Q} \).
Theorem 5.22 (Girsanov’s theorem)

Assume that the probability measures $\mathbb{P}$ and $\mathbb{Q}$ are mutually absolutely continuous on $\mathcal{F}_\infty$. Let $(D_t)_{t \geq 0}$ be the martingale with cadlag sample paths such that, for every $t \geq 0$,

$$D_t = \frac{d\mathbb{Q}}{d\mathbb{P}} \bigg|_{\mathcal{F}_t}.$$

Assume that $D$ has continuous sample paths, and let $L$ be the unique continuous local martingale such that $D_t = \mathcal{E}(L)_t$. Then, if $M$ is a continuous local martingale under $\mathbb{P}$, the process

$$\widetilde{M} = M - \langle M, L \rangle$$

is a continuous local martingale under $\mathbb{Q}$. 
Remark
By consequences of the martingale representation theorem explained at the end of the previous section, the continuity assumption for the sample paths of $D$ always holds when $(\mathcal{F}_t)$ is the (completed) canonical filtration of a Brownian motion. In applications of Theorem 5.22, one often starts from the martingale $(D_t)$ to define the probability measure $\mathbb{Q}$, so that the continuity assumption is satisfied by construction.

Proof of Theorem 5.22
The fact that $D_t$ can be written in the form $D_t = \mathcal{E}(L)_t$ follows from Proposition 5.21 (we are assuming that $D$ has continuous sample paths, and we also know from Proposition 5.20 that $D$ takes positive values). Then, let $T$ be a stopping time and let $X$ be an adapted process with continuous sample paths.
Remark
By consequences of the martingale representation theorem explained at the end of the previous section, the continuity assumption for the sample paths of $D$ always holds when $(\mathcal{F}_t)$ is the (completed) canonical filtration of a Brownian motion. In applications of Theorem 5.22, one often starts from the martingale $(D_t)$ to define the probability measure $\mathbb{Q}$, so that the continuity assumption is satisfied by construction.

Proof of Theorem 5.22
The fact that $D_t$ can be written in the form $D_t = \mathcal{E}(L)_t$ follows from Proposition 5.21 (we are assuming that $D$ has continuous sample paths, and we also know from Proposition 5.20 that $D$ takes positive values). Then, let $T$ be a stopping time and let $X$ be an adapted process with continuous sample paths.
Proof of Theorem 5.22 (cont)

We claim that, if \((XD)^T\) is a martingale under \(\mathbb{P}\), then \(X^T\) is a martingale under \(\mathbb{Q}\). Let us verify the claim. By Proposition 5.20, 
\[
\mathbb{E}_\mathbb{Q}[|X_{T\wedge t}|] = \mathbb{E}_\mathbb{P}[|X_{T\wedge t}|D_{T\wedge t}] < \infty,
\]
and it follows that \(X^T_t \in L^1(\mathbb{Q})\).
Then, let \(A \in \mathcal{F}_s\) and \(s < t\). Since \(A \cap \{T > s\} \in \mathcal{F}_s\), we have, using the fact that \((XD)^T\) is a martingale under \(\mathbb{P}\),
\[
\mathbb{E}_\mathbb{P}[1_{A \cap \{T > s\}} X_{T\wedge t}D_{T\wedge t}] = \mathbb{E}_\mathbb{P}[1_{A \cap \{T > s\}} X_{T\wedge s}D_{T\wedge s}].
\]
By Proposition 5.20,
\[
D_{T\wedge t} = \left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_{T\wedge t}}, \quad D_{T\wedge s} = \left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_{T\wedge s}}
\]
and thus, since \(A \cap \{T > s\} \in \mathcal{F}_{T\wedge s} \subset \mathcal{F}_{T\wedge t}\), it follows that
\[
\mathbb{E}_\mathbb{Q}[1_{A \cap \{T > s\}} X_{T\wedge t}] = \mathbb{E}_\mathbb{Q}[1_{A \cap \{T > s\}} X_{T\wedge s}].
\]
On the other hand, it is immediate that

\[ E_Q[1_{A \cap \{ T \leq s \}} X_{T \wedge t}] = E_Q[1_{A \cap \{ T \leq s \}} X_{T \wedge s}] \].

By combining with the previous display, we have

\[ E_Q[1_A X_{T \wedge t}] = E_Q[1_A X_{T \wedge s}] \], giving our claim. As a consequence of the claim, we get that, if \( XD \) is a continuous local martingale under \( P \), then \( X \) is a continuous local martingale under \( Q \).

Next let \( M \) be a continuous local martingale under \( P \), and let \( \tilde{M} \) be as in the statement of the theorem. We apply the preceding observation to \( X = \tilde{M} \), noting that, by Ito’s formula,
5.5 Girsanov’s Theorem

Proof of Theorem 5.22 (cont)

\[ \tilde{M}_t D_t = M_0 D_0 + \int_0^t \tilde{M}_s dD_s - \int_0^t D_s d\langle M, L \rangle_s + \langle M, D \rangle_t \]

\[ = M_0 D_0 + \int_0^t \tilde{M}_s dD_s + \int_0^t D_s dM_s \]

since \( d\langle M, L \rangle_s = D_s^{-1} d\langle M, D \rangle_s \) by Proposition 5.21. We get that \( \tilde{M} D \) is a continuous local martingale under \( \mathbb{P} \), and thus \( \tilde{M} \) is a continuous local martingale under \( \mathbb{Q} \).
Consequences

(a) A process $M$ which is a continuous local martingale under $\mathbb{P}$ remains a semimartingale under $\mathbb{Q}$, and its canonical decomposition under $\mathbb{Q}$ is $M = \tilde{M} + \langle M, L \rangle$ (recall that the notion of a finite variation process does not depend on the underlying probability measure). It follows that the class of semimartingales under $\mathbb{P}$ is contained in the class of semimartingales under $\mathbb{Q}$.

In fact these two classes are equal. Indeed, under the assumptions of Theorem 5.22, $\mathbb{P}$ and $\mathbb{Q}$ play symmetric roles, since the Radon-Nikodym derivative of $\mathbb{P}$ with respect to $\mathbb{Q}$ on the $\sigma$-field $\mathcal{F}_t$ is $D_t^{-1}$, which has continuous sample paths if $D$ does.

Furthermore

$$D_t^{-1} = \exp \left( -L_t + \langle L, L \rangle_t - \frac{1}{2} \langle L, L \rangle_t \right) = \exp \left( -\tilde{L}_t - \frac{1}{2} \langle \tilde{L}, \tilde{L} \rangle_t \right) = \mathcal{E}(\tilde{L})_t$$

where $\tilde{L} = L - \langle L, L \rangle$ is a continuous local martingale under $\mathbb{Q}$, and $\langle \tilde{L}, \tilde{L} \rangle = \langle L, L \rangle$. So, under the assumptions of Theorem 5.22, the roles of $\mathbb{P}$ and $\mathbb{Q}$ can be interchanged provided $D$ is replaced by $D^{-1}$ and $L$ is replaced by $-\tilde{L}$. 
(b) Let $X$ and $Y$ be two semimartingales (under $\mathbb{P}$ or under $\mathbb{Q}$). The bracket $\langle X, Y \rangle$ is the same under $\mathbb{P}$ and under $\mathbb{Q}$. In fact this bracket is given in both cases by the approximation of Proposition 4.21 (this observation was used implicitly in (a) above).

Similarly, if $H$ is a locally bounded progressive process, the stochastic integral $H \cdot X$ is the same under $\mathbb{P}$ and under $\mathbb{Q}$. To see this it is enough to consider the case when $X = M$ is a continuous local martingale (under $\mathbb{P}$). Write $(H \cdot M)_\mathbb{P}$ for the stochastic integral under $\mathbb{P}$ and $(H \cdot M)_\mathbb{Q}$ for the one under $\mathbb{Q}$. By linearity,

$$
(H \cdot \tilde{M})_\mathbb{P} = (H \cdot M)_\mathbb{P} - H \cdot \langle M, L \rangle = (H \cdot M)_\mathbb{P} - \langle (H \cdot M)_\mathbb{P}, L \rangle
$$

and Theorem 5.22 shows that $(H \cdot \tilde{M})_\mathbb{P}$ is a continuous local martingale under $\mathbb{Q}$. Furthermore the bracket of this continuous local martingale with any continuous local martingale $N$ under $\mathbb{Q}$ is equal to $H \cdot \langle M, N \rangle = H \cdot \langle \tilde{M}, N \rangle$ and it follows from Theorem 5.6 that $(H \cdot \tilde{M})_\mathbb{P} = (H \cdot \tilde{M})_\mathbb{Q}$ and hence also $(H \cdot M)_\mathbb{P} = (H \cdot M)_\mathbb{Q}$.
With the notation of Theorem 5.22, set $\tilde{M} = \mathcal{G}_Q^P(M)$. Then $\mathcal{G}_Q^P$ maps the set of all $P$-continuous local martingales onto the set of all $Q$-continuous local martingales. One easily verifies, using the remarks in (a) above, that $\mathcal{G}_Q^P \circ \mathcal{G}_Q^P = Id$. Furthermore, the mapping $\mathcal{G}_Q^P$ commutes with the stochastic integral: if $H$ is a locally bounded progressive process, $H \cdot \mathcal{G}_Q^P(M) = \mathcal{G}_Q^P(H \cdot M)$.

(c) Suppose that $M = B$ is an $(\mathcal{F}_t)$-Brownian motion under $P$, then $\tilde{B} = B - \langle B, L \rangle$ is a continuous local martingale under $Q$, with quadratic variation $\langle \tilde{B}, \tilde{B} \rangle_t = \langle B, B \rangle_t = t$. By Theorem 5.12, it follows that $\tilde{B}$ is an $(\mathcal{F}_t)$-Brownian motion under $Q$. 