Math 562 Fall 2020

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October 21, 2020

Outline

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1. 5.4 The Representation of Martingales as Stochastic Integrals

In the special setting where the filtration on Ω is the completed canonical filtration of a Brownian motion, we will now show that all martingales can be represented as stochastic integrals with respect to that Brownian motion. For the sake of simplicity, we first consider a one-dimensional Brownian motion. We will discuss the extension to Brownian motion in higher dimensions at the end of this section.

Theorem 5.18

Assume that the filtration $(\mathcal{F}_t)$ on Ω is the completed canonical filtration of a 1-dimensional Brownian motion B started from 0. Then, for every random variable $Z \in L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$, there exists a unique progressive process $h \in L^2(B)$ (i.e. $\mathbb{E}[\int_0^\infty h_s^2 ds] < \infty$) such that

$$Z = \mathbb{E}[Z] + \int_0^\infty h_s dB_s.$$

Consequently, for every martingale M that is bounded in L^2 (respectively, for every continuous local martingale M), there exists a unique process $h \in L^2(B)$ (reps. $h \in L^2_{loc}(B)$) and a constant $C \in \mathbb{R}$ such that

$$M_t = C + \int_0^t h_s dB_s.$$

Remark

As the proof will show, the second part of the statement applies to a martingale M that is bounded in L^2, without any assumption on the continuity of sample paths of M. This observation will be useful later when we discuss consequences of the representation theorem. Note that continuous local martingales have continuous sample paths by definition.

Lemma 5.19

Under the assumptions of the theorem, the vector space generated by the random variables

$$\exp\left(i\sum_{j=1}^{n}\lambda_j(B_{t_j}-B_{t_{j-1}})\right), \quad n\geq 1, 0=t_0<t_1<\cdots<t_n, \lambda_1,\ldots,\lambda_n\in\mathbb{R},$$

is dense in the space $L^2_{\mathbb{C}}(\Omega,\mathcal{F}_\infty,\mathbb{P})$ of all square-integrable complex-valued $\mathcal{F}_\infty$-measurable random variables.

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Proof of Lemma 5.19

It suffices to show that, if $Z \in L^2_{\mathbb{C}}(\Omega, \mathcal{F}_\infty, \mathbb{P})$ is such that

$$\mathbb{E}\left[Z \exp\left(i \sum_{j=1}^{n} \lambda_j (B_{t_j} - B_{t_{j-1}}) \right) \right] = 0 \tag{1}$$

for all $n \geq 1$, $0 = t_0 < t_1 < \cdots < t_n$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, then $Z = 0$.

Fix $0 = t_0 < t_1 < \cdots < t_n$, and consider the complex measure μ on $\mathbb{R}^n$ defined by

$$\mu(F) = \mathbb{E}[Z 1_F(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})], \quad F \in \mathcal{B}(\mathbb{R}^n).$$

Then (1) exactly shows that the Fourier transform of μ is identically zero. By the injectivity of the Fourier transform on complex measures on $\mathbb{R}^n$, it follows that $\mu = 0$. We have thus $\mathbb{E}[Z 1_A] = 0$ for every $A \in \sigma(B_{t_1}, \ldots, B_{t_n})$.

Proof of Lemma 5.19 (cont)

A monotone class argument then shows that the identity $\mathbb{E}[Z1_A] = 0$ remains valid for any $A \in \sigma(B_t : t \geq 0)$, and then by completion for any $A \in \mathcal{F}_\infty$. It follows that $Z = 0$.

Proof of Theorem 5.18

We start with the first assertion. We first observe that the uniqueness of h is easy since, if the representation of a given variable Z holds with two processes h and h' in $L^2(B)$, we have

$$\mathbb{E}[\int_0^\infty (h_s - h'_s)^2 ds] = \mathbb{E}\left[\left(\int_0^\infty h_s dB_s - \int_0^\infty h'_s dB_s\right)^2\right] = 0,$$

hence $h = h'$ in $L^2(B)$.

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Proof of Theorem 5.18 (cont)

Let us turn to the existence part. Let $\mathcal{H}$ stand for the vector space of all variables $Z \in L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$ for which the property of the statement holds. We note that if $Z \in \mathcal{H}$ and h is the associated process in $L^2(B)$, we have

$$\mathbb{E}[Z^2] = (\mathbb{E}[Z])^2 + \mathbb{E}[\int_0^\infty h_s^2 ds].$$

It follows that $\mathcal{H}$ is a closed subspace of $L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$. Indeed, if (Z_n) is a sequence in $\mathcal{H}$ that converges to Z in $L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$, the processes $h^{(n)}$ corresponding respectively to the variables Z_n form a Cauchy sequence $L^2(B)$, hence converge in $L^2(B)$ to a certain process $h \in L^2(B)$ (here we use the Hilbert space structure of $L^2(B)$), and it immediately follows that $Z = \mathbb{E}[Z] + \int_0^\infty h_s dB_s$.

Proof of Theorem 5.18 (cont)

Since $\mathcal{H}$ is closed, in order to prove that $\mathcal{H} = L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$, we just have to verify that $\mathcal{H}$ contains a dense subset of $L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$. Let $0 = t_0 < t_1 < \cdots < t_n$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, define $f(s) = \sum_{j=1}^n \lambda_j 1_{(t_{j-1}, t_j]}(s)$. Write $\mathcal{E}_t^f$ for the exponential martingale $\mathcal{E}(i \int_0^t f(s) dB_s)$. Proposition 5.11 shows that

$$\exp\left(i \sum_{j=1}^n \lambda_j (B_{t_j} - B_{t_{j-1}}) + \frac{1}{2} \sum_{j=1}^n \lambda_j^2 (t_j - t_{j-1}) \right) = \mathcal{E}_\infty^f = 1 + i \int_0^\infty \mathcal{E}_s^f f(s) dB_s$$

and it follows that both the real part and the imaginary part of variables of the form $\exp(i \sum_{j=1}^n \lambda_j (B_{t_j} - B_{t_{j-1}}))$ are in $\mathcal{H}$. By Lemma 5.19, linear combinations of such random variables are dense in $L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$. This completes the proof of the first assertion of the theorem.

Proof of Theorem 5.18 (cont)

Let us turn to the second assertion. If M is a martingale that is bounded in L^2, then $M_\infty \in L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$, and thus can be written in the form

$$M_\infty = \mathbb{E}[M_\infty] + \int_0^\infty h_s dB_s$$

where $h \in L^2(B)$. Thus

$$M_t = \mathbb{E}[M_\infty | \mathcal{F}_t] = \mathbb{E}[M_\infty] + \int_0^t h_s dB_s$$

and the uniqueness of h is also immediate from the uniqueness in the first assertion.

Proof of Theorem 5.18 (cont)

Finally, if M is a continuous local martingale, we have first $M_0 = C \in \mathbb{R}$ since $\mathcal{F}_0$ is trivial. If $T_n = \inf\{t \geq 0 : |M_t| \geq n\}$, we can apply the case of martingales bounded in L^2 to M^{T_n} and we get a process $h^{(n)} \in L^2(B)$ such that

$$M_t^{T_n} = C + \int_0^t h_s^{(n)} dB_s.$$

Using the uniqueness of the progressive process in the representation, we get that $h_s^{(m)} = 1_{[0,T_m]} h_s^{(n)}$ if $m < n$, ds-a.e., a.s. It is now easy to construct a process $h \in L^2_{loc}(B)$ such that, for every m, $h_s^{(m)} = 1_{[0,T_m]} h_s$ if $m < n$, ds-a.e., a.s. The representation formula of the theorem follows, and the uniqueness of h is also straightforward.

Consequences of the representation theorem

(1) The filtration $(\mathcal{F}_t)$ is right-continuous. Indeed, let $t \geq 0$ and let Z be $\mathcal{F}_{t+}$-measurable and bounded.We can find $h \in L^2(B)$ such that

$$Z = \mathbb{E}[Z] + \int_0^\infty h_s dB_s.$$

If $\epsilon > 0$, Z is $\mathcal{F}_{t+\epsilon}$-measurable and thus

$$Z = \mathbb{E}[Z|\mathcal{F}_{t+\epsilon}] = \mathbb{E}[Z] + \int_0^{t+\epsilon} h_s dB_s.$$

When $\epsilon \to 0$ the right-hand side converges in L^2 to

$$\mathbb{E}[Z] + \int_0^t h_s dB_s$$

Thus Z is equal a.s. to an $\mathcal{F}_t$-measurable random variable, and, since the filtration is complete, Z is $\mathcal{F}_t$-measurable.

A similar argument shows that the filtration $(\mathcal{F}_t)$ is also left-continuous: If, for $t > 0$, we let

$$\mathcal{F}_{t-} = \sigma(\cup_{s<t}\mathcal{F}_s),$$

we have $\mathcal{F}_{t-} = \mathcal{F}_t$.
(2) All martingales wrt the filtration $(\mathcal{F}_t)$ have a modification with continuous sample paths. For a martingale that is bounded in L^2, this follows from the representation formula (see the remark after the statement of the theorem). Then consider a uniformly integrable martingale M (if M is not uniformly integrable, we just replace M by $M_{t\wedge a}$ for every $a > 0$). In this case, we have, for every $t \geq 0$,

$$M_t = \mathbb{E}[M_\infty|\mathcal{F}_t].$$

By Theorem 3.18 (whose application is justified as we know that the filtration is right-continuous), the process M_t has a modification with cadlag sample paths, and we consider this modification. Let $M_\infty^{(n)}$

be a sequence of bounded random variables such that $M_\infty^{(n)} \to M_\infty$ in L^1 as $n \to \infty$. Introduce the martingales

$$M_t^{(n)} = \mathbb{E}[M_\infty^{(n)} | \mathcal{F}_t],$$

which are bounded in L^2. By the beginning of the argument, we can assume that, for every $n \geq 1$, the sample paths of $M^{(n)}$ are continuous. On the other hand, Doob's maximal inequality implies that, for every $\lambda > 0$,

$$\mathbb{P}\left[\sup_{t\geq 0} |M_t^{(n)} - M_t| > \lambda\right] \leq \frac{3}{\lambda}\mathbb{E}[|M_\infty^{(n)} - M_\infty|].$$

It follows that we can find a sequence $n_k \uparrow \infty$ such that, for every $k \geq 1$,

$$\mathbb{P}\left[\sup_{t\geq 0} |M_t^{(n_k)} - M_t| > \lambda\right] \leq 2^{-k}.$$

An application of the Borel-Cantelli lemma now shows that

$$\sup_{t\geq 0} |M_t^{(n_k)} - M_t| \to 0, \quad a.s \text{ as } k \uparrow \infty$$

and we get that the sample paths of M are continuous as uniform limits of continuous functions.

Multidimensional extension

Let us briefly describe the multidimensional extension of the preceding results. We now assume that the filtration $(\mathcal{F}_t)$ on Ω is the completed canonical filtration of a d-dimensional Brownian motion $B = (B^1, \ldots, B^d)$ started at 0. Then, for every random variable $Z \in L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$, there exists a unique d-tuple $(h^1, \ldots, h^d)$ of progressive processes, satisfying

$$\mathbb{E}\left[\int_0^\infty (h_s^j)^2 ds\right] < \infty, \quad j = 1, \ldots, d$$

such that

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$$\mathbb{E}\left[\int_0^\infty (h_s^j)^2 ds\right] < \infty, \quad j = 1, \dots, d$$

such that

$$Z = \mathbb{E}[Z] + \sum_{j=1}^{d} \int_0^\infty h_s^l dB_s^j.$$

The proofs are exactly the same as in the case $d = 1$. Consequences (1) and (2) above remain valid.