Math 562 Fall 2020

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October 19, 2020

Outline

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Recall that for any continuous local martingale M, we define

$$M_t^* = \sup_{s \le t} |M_s|, \quad t \ge 0.$$

Theorem 5.16 (Burkholder-Davis-Gundy inequalities)

For every $p > 0$, there exist two constants $c_p, C_p > 0$ depending only on p such that, for every continuous local martingale M with $M_0 = 0$, and every stopping time T,

$$c_p \mathbb{E}[\langle M, M \rangle_T^{p/2}] \le \mathbb{E}[(M_T^*)^p] \le C_p \mathbb{E}[\langle M, M \rangle_T^{p/2}].$$

Remark

It may happen that both $\mathbb{E}[\langle M, M \rangle_T^{p/2}]$ and $\mathbb{E}[(M_T^*)^p]$ are infinite. The theorem says that these quantities are either both finite (then the stated bounds hold) or both infinite.

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Proof of Theorem 5.16

Replacing M by the stopping martingale M^T, we see that it is enough to treat the special case $T = \infty$. We then observe that it suffices to consider the case when M is bounded: Assuming that the bounded case has been treated, we can replace M by M^{T_n}, where $T_n = \inf\{t \geq 0 : |M_t| = n\}$, and we get the general case by letting $n \to \infty$.

(1) $p \geq 2$, right-hand inequality: Apply Ito's formula to the function $|x|^p$:

$$|M_t|^p = \int_0^t p|M_s|^{p-1}\mathrm{sgn}(M_s)dM_s + \frac{1}{2}\int_0^t p(p-1)|M_s|^{p-2}d\langle M, M\rangle_s.$$

Since M is bounded, the process

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Proof of Theorem 5.16 (cont)

$$\mathbb{E}[|M_t|^p] = \frac{p(p-1)}{2}\mathbb{E}\left[\int_0^t |M_s|^{p-2} d\langle M, M\rangle_s\right]$$
$$\leq \frac{p(p-1)}{2}\mathbb{E}[(M_t^*)^{p-2}\langle M, M\rangle_t]$$
$$\leq \frac{p(p-1)}{2}(\mathbb{E}[(M_t^*)^p])^{(p-2)/2}(\mathbb{E}[\langle M, M\rangle_t^{p/2}])^{2/p}$$

by Hölder's inequality. On the other hand, by Doob's L^p inequality,

$$\mathbb{E}[(M_t^*)^p] \leq (\frac{p}{p-1})^p \mathbb{E}[|M_t|^p]$$

and combining this bound with the previous one, we arrive at

$$\mathbb{E}[(M_t^*)^p] \leq \left((\frac{p}{p-1})^p \frac{p(p-1)}{2}\right)^{p/2} \mathbb{E}[\langle M, M\rangle_t^{p/2}].$$

It now suffices to let t tend to ∞.

Proof of Theorem 5.16 (cont)

(2) $p \geq 4$, left-hand inequality: For any $q \geq 2$, there exists $a_q > 0$ such that

$$|x+y|^q \leq a_q(|x|^q + |y|^q), \quad x, y \in \mathbb{R}.$$

Since $M_t^2 = 2\int_0^t M_s dM_s + \langle M, M\rangle_t$, we have

$$\mathbb{E}[\langle M, M\rangle_\infty^{p/2}] \leq a_p \left(\mathbb{E}[(M_\infty^*)^p] + \mathbb{E}[|\int_0^\infty M_s dM_s|^{p/2}] \right).$$

Applying (1) to $\int_0^\cdot M_s dM_s$, we get

$$\begin{aligned}\mathbb{E}[\langle M, M\rangle_\infty^{p/2}] &\leq a_p \left(\mathbb{E}[(M_\infty^*)^p] + \mathbb{E}\left[\left(\int_0^\infty M_s^2 d\langle M, M\rangle_s \right)^{p/4} \right] \right) \\ &\leq a_p \left(\mathbb{E}[(M_\infty^*)^p] + \left(\mathbb{E}[(M_\infty^*)^p]\mathbb{E}[\langle M, M\rangle_\infty^{p/2}] \right)^{1/2} \right).\end{aligned}$$

Proof of Theorem 5.16 (cont)

If we let

$$x = \mathbb{E}[\langle M, M\rangle_\infty^{p/2}]^{1/2}, \quad y = \mathbb{E}[(M_\infty^*)^p]^{1/2},$$

the above inequality reads

$$x^2 - a_p xy - a_p y^2 \leq 0,$$

which forces x to be less than or equal to the positive root of $x^2 - a_p xy - a_p y^2 = 0$, which is of the form $a_p y$. This finishes the proof of this part.

(3) $p < 2$, right-hand inequality: Since $M \in \mathbb{H}^2$, $M^2 - \langle M, M\rangle$ is a uniformly integrable martingale and we have, for every stopping time T,

$$\mathbb{E}[M_T^2] = \mathbb{E}[\langle M, M\rangle_T].$$

Let $x > 0$ and consider the stopping time $T_x = \inf\{t \geq 0 : M_t^2 \geq x\}$.

Proof of Theorem 5.16 (cont)

Then, if T is any bounded stopping time,

$$\mathbb{P}((M_T^*)^2 \geq x) = \mathbb{P}(T_x \leq T) = \mathbb{P}((M_{T_x \wedge T}^*)^2 \geq x)$$
$$\leq \frac{1}{x}\mathbb{E}[(M_{T_x \wedge T}^*)^2] = \frac{1}{x}\mathbb{E}[\langle M, M\rangle_{T_x \wedge T}] \leq \frac{1}{x}\mathbb{E}[\langle M, M\rangle_T].$$

Next consider the stopping time $S_x = \inf\{t \geq 0 : \langle M, M\rangle_t \geq x\}$. Note

$$\{(M_t^*)^2 \geq x\} \subset \{(M_{S_x \wedge t}^*)^2 \geq x\} \cup \{S_x \leq t\}, \quad t \geq 0.$$

Using the preceding bound with $T = S_x \wedge t$, we get

$$\mathbb{P}((M_t^*)^2 \geq x) \leq P((M_{S_x \wedge t}^*)^2 \geq x) + P(S_x \leq t)$$
$$\leq \frac{1}{x}E[\langle M, M\rangle_{S_x \wedge t}] + P(\langle M, M\rangle_t \geq x)$$
$$= \frac{1}{x}E[\langle M, M\rangle_t \wedge x] + P(\langle M, M\rangle_t \geq x)$$
$$= \frac{1}{x}E[\langle M, M\rangle_t 1_{\langle M, M\rangle_t < x}] + 2P(\langle M, M\rangle_t \geq x).$$

Proof of Theorem 5.16 (cont)

To complete the proof, set $q = p/2 \in (0,1)$ and integrate each side of the last bound with respect to the measure $qx^{q-1}dx$. We have first

$$\int_0^\infty \mathbb{P}((M_t^*)^2 \geq x) qx^{q-1} dx = \mathbb{E}[\int_0^{(M_t^*)^2} qx^{q-1} dx] = \mathbb{E}[(M_t^*)^{2q}]$$

and similarly

$$\int_0^\infty P(\langle M, M\rangle_t \geq x) qx^{q-1} dx = E[\langle M, M\rangle_t^q].$$

Proof of Theorem 5.16 (cont)

Furthermore,

$$\int_0^\infty \frac{1}{x}\mathbb{E}[\langle M, M\rangle_t 1_{\langle M,M\rangle_{<x}}]x^{q-1}dx$$
$$=\mathbb{E}[\langle M, M\rangle_t \int_{\langle M,M\rangle_t}^\infty qx^{q-2}dx] = \frac{q}{1-q}\mathbb{E}[\langle M, M\rangle_t^q].$$

Summarizing, we have

$$\mathbb{E}[(M_t^*)^{2q}] \leq \left(2 + \frac{q}{1-q}\right)\mathbb{E}[\langle M, M\rangle_t^q].$$

Letting $t \uparrow \infty$, we get the desired result in this step.

Definition

A positive, adapted right-continuous process $X = (X_t)_{t \geq 0}$ is said to be dominated by an increasing process $A = (A_t)_{t \geq 0}$, if

$$\mathbb{E}[X_T | \mathcal{F}_0] \leq \mathbb{E}[A_T | \mathcal{F}_0]$$

for every bounded stopping time T. Here A may not be continuous and A_0 may not be zero (different from the usual meaning of increasing processes in the textbook).

Lemma

If X is a positive adapted right-continuous process dominated by an increasing process A and A is continuous, then for any $x, y > 0$,

$$\mathbb{P}(X_\infty^* > x, A_\infty \leq y) \leq \frac{1}{x}\mathbb{E}[A_\infty \wedge y].$$

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Lemma

If X is a positive adapted right-continuous process dominated by an increasing process A and A is continuous, then for any $x, y > 0$,

$$\mathbb{P}(X^*_\infty > x, A_\infty \leq y) \leq \frac{1}{x}\mathbb{E}[A_\infty \wedge y].$$

Proof

It suffices to prove the inequality in the case where $\mathbb{P}(A_0 \le y) > 0$ and, in fact, even $P(A_0 \le y) = 1$, which may be achieved by replacing $\mathbb{P}$ by $\mathbb{P}^*(\cdot) = P(\cdot | A_0 \le y)$ under which the domination relation is still satisfied.

Moreover, by Fatou's lemma, it is enough to prove that

$$\mathbb{P}(X_n^* > x, A_n \le y) \le \frac{1}{x}\mathbb{E}[A_n \wedge y], \quad n \ge 1.$$

But reasoning on $[0, n]$ amounts to reasoning on $[0, \infty]$ and assuming that the random variable X_∞ exists and the domination relation is true for all stopping times, whether bounded or not. Define

$$R = \inf\{t \ge 0 : A_t > y\}, \quad S = \inf\{t \ge 0 : X_t > x\}.$$

Proof

$\{A_\infty \le y\} \subset \{R = \infty\}$ and consequently

$$\begin{aligned}
\mathbb{P}(X^*_\infty > x, A_\infty \le y) &= \mathbb{P}(X^*_\infty > x, R + \infty) \\
&\le \mathbb{P}(X_S \ge x, S < \infty, R = \infty) \\
&\le \mathbb{P}(X_{S\wedge R} \ge x) \le \frac{1}{x}\mathbb{E}[X_{S\wedge R}] \\
&\le \frac{1}{x}\mathbb{E}[A_{S\wedge R}] \le \frac{1}{x}\mathbb{E}[A_\infty \wedge y],
\end{aligned}$$

the last inequality being satisfied since, thanks to the continuity of A, and $A_0 \le y$ a.s., we have $A_{S\wedge R} \le A_\infty \wedge y$.

Proposition

Under the assumptions of the lemma above, for any $k \in (0, 1)$,

$$\mathbb{E}[(X_\infty^*)^k] \leq \frac{2-k}{1-k}\mathbb{E}[A_\infty^k].$$

Proof

Let $F : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and $F(0) = 0$. By Fubini and the lemma above,

$$\begin{aligned}
\mathbb{E}[F(X_\infty^*)] &= \mathbb{E}[\int_0^\infty 1_{\{X_\infty^* > x\}} dF(x)] \\
&\leq \int_0^\infty \left(\mathbb{P}(X_\infty^* > x, A_\infty \leq x) + \mathbb{P}(A_\infty > x)\right) dF(x) \\
&\leq \int_0^\infty \left(\frac{1}{x}\mathbb{E}[A_\infty \wedge x] + \mathbb{P}(A_\infty > x)\right) dF(x) \\
&\leq \int_0^\infty \left(2\mathbb{P}(A_\infty > x) + \frac{1}{x}\mathbb{E}[A_\infty 1_{\{A_\infty \leq x\}}]\right) dF(x)
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$$\mathbb{E}[F(X^*_\infty)] = \mathbb{E}[\int_0^\infty 1_{\{X^*_\infty > x\}} dF(x)]$$
$$\leq \int_0^\infty (\mathbb{P}(X^*_\infty > x, A_\infty \leq x) + \mathbb{P}(A_\infty > x))\, dF(x)$$
$$\leq \int_0^\infty \left(\frac{1}{x}\mathbb{E}[A_\infty \wedge x] + \mathbb{P}(A_\infty > x)\right) dF(x)$$
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Proof

$$= 2\mathbb{E}[F(A_\infty)] + \mathbb{E}[A_\infty \int_{A_\infty}^{\infty} \frac{dF(x)}{x}]$$

$$= \mathbb{E}[\widetilde{F}(A_\infty)]$$

where $\widetilde{F}(x) = 2F(x) + x\int_x^\infty \frac{dF(u)}{u}$. Take $F(x) = x^k$, we obtain the desired result.

Proof of Theorem 5.16 (cont)

Take $X_t = (M_t^*)^2$ and $A_t = C_2\langle M, M\rangle_t$ for the right-hand inequality, and $X_t = \langle M, M\rangle_t^2$ and $A_t = \frac{1}{c_4}(M_t^*)^4$ for the left-hand inequality. The necessary domination relations follow from steps (1) and (2).

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