Math 562 Fall 2020

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Outline

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Theorem 5.13 (Dambis-Dubins-Schwarz)

Let M be a continuous local martingale such that $\langle M, M\rangle_\infty = \infty$ a.s. There exists a Brownian motion $(\beta_s)_{s\geq 0}$ such that

$$a.s. \quad \forall t \geq 0, \quad M_t = \beta_{\langle M,M\rangle_t}.$$

Remarks

(i) One can remove the assumption $\langle M, M\rangle_\infty = \infty$, at the cost of enlarging the underlying probability space.

(ii) The Brownian motion β is not adapted with respect to the filtration $(\mathcal{F}_t)$, but with respect to a "time-changed" filtration, as the proof will show.

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Proof of Theorem 5.13

We first assume that $M_0 = 0$. For every $r \geq 0$, we define

$$\tau_r = \inf\{t \geq 0 : \langle M, M\rangle_t \geq r\}.$$

τ_r is a stopping time. $\tau_r < \infty$ for every $r \geq 0$ on the event $\{\langle M, M\rangle_\infty = \infty\}$. It will be convenient to redefine τ_r on the (negligible) event $\mathcal{N}\{\langle M, M\rangle_\infty < \infty\}$ by setting $\tau_r(\omega) = 0$ for every $r \geq 0$ if $\omega \in \mathcal{N}$. Since the filtration is complete, τ_r remains a stopping time after this modification.

By construction, for every $\omega \in \Omega$, the function $r \mapsto \tau_r(\omega)$ is non-decreasing and left-continuous, and therefore has a right limit at every $r \geq 0$. Denote this right limit by τ_{r+}. Then we have

$$\tau_{r+} = \inf\{t \geq 0 : \langle M, M\rangle_t > r\}$$

except of course on the negligible set $\mathcal{N}$, where $\tau_{r+} = 0$.

Proof of Theorem 5.13 (cont)

We define $\beta_r = M_{\tau_r}$ for every $r \geq 0$. By Theorem 3.7, $(\beta_r)_{r\geq 0}$ is adapted to $(\mathcal{G}_r)$ where $\mathcal{G}_r = \mathcal{F}_{\tau_r}$ for every $r \geq 0$ and $\mathcal{G}_\infty = \mathcal{F}_\infty$. The filtration $(\mathcal{G}_r)$ is complete since $(\mathcal{F}_t)$ is complete.

The sample paths $r \mapsto \beta_r(\omega)$ are left-continuous and have right limits given for every $r \geq 0$ by

$$\beta_{r+} = \lim_{s\downarrow\downarrow r} \beta_s = M_{\tau_{r+}}.$$

In fact we have $\beta_{r+} = \beta_r$ for every $r \geq 0$ by the following lemma:

Lemma 5.14

We have a.s. for every $0 \leq a < b$,

$$M_t = M_a, \forall t \in [a,b] \Leftrightarrow \langle M, M\rangle_b = \langle M, M\rangle_a.$$

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Proof of Theorem 5.13 (cont)

Since $\langle M, M\rangle_{\tau_r} = \langle M, M\rangle_{\tau_{r+}}$ for every $r \geq 0$, Lemma 5.14 implies that $M_{\tau_r} = M_{\tau_{r+}}$ for every $r \geq 0$ a.s. Hence the sample paths of β are continuous (to be precise, we should redefine $\beta_r = 0$, for every $r \geq 0$, on the negligible set where the property of Lemma 5.14 fails).

We now show that β_s and $\beta_s^2 - s$ are martingales with respect to the filtration $(\mathcal{G}_s)$. For every integer $n \geq 1$, the stopped continuous local martingales M^{τ_n} and $(M^{\tau_n})^2 - \langle M, M\rangle^{\tau_n}$ are uniformly integrable martingales (by Theorem 4.13, recalling that $M_0 = 0$ and noting that $\langle M^{\tau_n}, M^{\tau_n}\rangle_\infty = \langle M, M\rangle^{\tau_n} = n$ a.s.) The optional stopping theorem then implies that, for every $0 \leq r \leq s \leq n$,

$$\mathbb{E}[\beta_s|\mathcal{G}_r] = \mathbb{E}[M^{\tau_n}_{\tau_s}|\mathcal{F}_{\tau_r}] = M^{\tau_n}_{\tau_r} = \beta_r$$

and similarly

$$\mathbb{E}[\beta_s^2 - s|\mathcal{G}_r] = \mathbb{E}[(M^{\tau_n}_{\tau_s})^2 - \langle M^{\tau_n}, M^{\tau_n}\rangle_{\tau_s}|\mathcal{F}_{\tau_r}] = (M^{\tau_n}_{\tau_r})^2 - \langle M^{\tau_n}, M^{\tau_n}\rangle_{\tau_r} = \beta_r^2 - r.$$

Proof of Theorem 5.13 (cont)

Theorem 5.12 shows that β is a $(\mathcal{G}_r)$-Brownian motion. Finally, by the definition of β, we have a.s. for every $t \geq 0$,

$$\beta_{\langle M,M\rangle_t} = M_{\tau_{\langle M,M\rangle_t}}.$$

But since $\tau_{\langle M,M\rangle_t} \leq t \leq \tau_{\langle M,M\rangle_{t+}}$ and since $\langle M, M\rangle$ takes the same value at $\tau_{\langle M,M\rangle_t}$ and at $\tau_{\langle M,M\rangle_{t+}}$, Lemma 5.14 shows that $M_t = M_{\tau_{\langle M,M\rangle_t}}$ for every $t \geq 0$ a.s. We conclude that we have $M_t = \beta_{\langle M,M\rangle_t}$ for every $t \geq 0$ a.s. This completes the proof when $M_0 = 0$.

If $M_0 \neq 0$, we write $M_t = M_0 + M'_t$, and we apply the previous argument to M' to get a Brownian motion β' with $\beta'_0 = 0$, such that $M'_t = \beta'_{\langle M',M'\rangle_t}$ for every $t \geq 0$ a.s. Since β' is a $(\mathcal{G}_r)$-Brownian motion, β' is independent of $\mathcal{G}_0 = \mathcal{F}_0$, hence of M_0. Therefore, $\beta_s = M_0 + \beta'_s$ is also a Brownian motion, and we get the desired representation for M.

Proof of Lemma 5.14

Thanks to the continuity of sample paths of M and $\langle M, M\rangle$, it suffices to show that for any fixed a and b such that $0 \le a \le b$, we have

$$\{M_t = M_a, \forall t \in [a, b]\} = \{\langle M, M\rangle_b = \langle M, M\rangle_a\}, \quad a.s.$$

The fact that the event in the left-hand side is (a.s.) contained in the event in the right-hand side is easy from the approximations of $\langle M, M\rangle$ in Theorem 4.9.

Let us prove the converse. Consider the continuous local martingale $N_t = M_t - M_{t\wedge a}$ and note that

$$\langle N, N\rangle_t = \langle M, M\rangle_t - \langle M, M\rangle_{t\wedge a}.$$

For every $\epsilon > 0$ define the stopping time

$$T_\epsilon = \inf\{t \ge 0 : \langle N, N\rangle_t \ge \epsilon\}.$$

Proof of Lemma 5.14 (cont)

Then N^{T_ϵ} is a martingale in $\mathbb{H}^2$ (since $\langle N^{T_\epsilon}, N^{T_\epsilon}\rangle_\infty \le \epsilon$). Fix $t \in [a, b]$. We have

$$\mathbb{E}[N^2_{t\wedge T_\epsilon}] = \mathbb{E}[\langle N, N\rangle_{t\wedge T_\epsilon}] \le \epsilon.$$

Hence, considering the event $A := \{\langle M, M\rangle_b = \langle M, M\rangle_a\} \subset \{T_\epsilon \ge b\}$,

$$\mathbb{E}[1_A N_t^2] = \mathbb{E}[1_A N^2_{t\wedge T_\epsilon}] \le \mathbb{E}[N^2_{t\wedge T_\epsilon}] \le \epsilon.$$

By letting ϵ go to 0, we get $\mathbb{E}[1_A N_t^2] = 0$ and thus $N_t = 0$ a.s. on A, which completes the proof.

Combining the arguments of the proof of Theorem 5.13 with Theorem 5.12, we get the following result.

Proof of Lemma 5.14 (cont)

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Combining the arguments of the proof of Theorem 5.13 with Theorem 5.12, we get the following result.

Proposition 5.15

Let M and N be two continuous local martingales such that $M_0 = N_0 = 0$. Assume that

(i) $\langle M, M\rangle_t = \langle N, N\rangle_t$ for every $t \geq 0$ a.s..

(ii) M and N are orthogonal ($\langle M, N\rangle_t = 0$ for every $t \geq 0$ a.s.).

(iii) $\langle M, M\rangle_\infty = \langle N, N\rangle_\infty$ a.s.

Let $\beta = (\beta_s)_{s\geq 0}$, resp. $\gamma = (\gamma_s)_{s\geq 0}$ be 1-dim Brownian motions such that $M_t = \beta_{\langle M,M\rangle_t}$, resp. $N_t = \gamma_{\langle N,N\rangle_t}$ for every $t \geq 0$ a.s.. Then β and γ are independent.

Proof of Proposition 5.15

We use the notation of the proof of Theorem 5.13 and note that we have $\beta_r = M_{\tau_r}$ and $\gamma_r = N_{\tau_r}$, where

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Let $\beta = (\beta_s)_{s\geq 0}$, resp. $\gamma = (\gamma_s)_{s\geq 0}$ be 1-dim Brownian motions such that $M_t = \beta_{\langle M,M\rangle_t}$, resp. $N_t = \gamma_{\langle N,N\rangle_t}$ for every $t \geq 0$ a.s.. Then β and γ are independent.

Proof of Proposition 5.15

We use the notation of the proof of Theorem 5.13 and note that we have $\beta_r = M_{\tau_r}$ and $\gamma_r = N_{\tau_r}$, where

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Proof of Proposition 5.15 (cont)

We know that β and γ are $(\mathcal{G}_r)$-Brownian motions. Since M and N are orthogonal martingales, we also know that $M_t N_t$ is a local martingale. As in the proof of Theorem 5.13, and using now Proposition 4.15 (v), we get that, for every $n \geq 1$, $M^{\tau_n} N^{\tau_n}$ is a uniformly integrable martingale, and by applying the optional stopping theorem, we obtain that for $r \leq s \leq n$,

$$\mathbb{E}[\beta_s \gamma_s | \mathcal{G}_r] = \mathbb{E}[M^{\tau_n}_{\tau_s} N^{\tau_n}_{\tau_s} | \mathcal{F}_{\tau_r}] = M^{\tau_n}_{\tau_r} N^{\tau_n}_{\tau_r} = \beta_r \gamma_r$$

so that $\beta_r \gamma_r$ is a $(\mathcal{G}_r)$-martingale and the bracket $\langle \beta, \gamma \rangle$ (evaluated in the filtration $(\mathcal{G}_r)$) is identically zero. By Theorem 5.12, it follows that (β, γ) is a 2-dim Brownian motion and, since $\beta_0 = \gamma_0 = 0$, this implies that β and γ are independent.

We now give some important inequalities connecting a continuous local martingale with its quadratic variation. If M is a continuous local martingale, we define

$$M_t^* = \sup_{s \leq t} |M_s|, \quad t \geq 0.$$

Theorem 5.16 below shows that, under the condition $M_0 = 0$, for every $p > 0$, the p-th moment of M_t^* is comparable to that of $\sqrt{\langle M, M \rangle_t}$. These bounds are very useful because, in particular when M is a stochastic integral, it is often easier to estimate the moments of $\langle M, M \rangle_t$ than those of M_t^*. Such applications arise, for instance, in the study of stochastic differential equations.

Theorem 5.16 (Burkholder-Davis-Gundy inequalities

For every $p > 0$, there exist two constants $c_p, C_p > 0$ depending only on p such that, for every continuous local martingale M with $M_0 = 0$, and every stopping time T,

$$c_p \mathbb{E}[\langle M, M\rangle_T^{p/2}] \leq \mathbb{E}[(M_T^*)^p] \leq C_p \mathbb{E}[\langle M, M\rangle_T^{p/2}].$$

Remark

It may happen that both $\mathbb{E}[\langle M, M\rangle_T^{p/2}]$ and $\mathbb{E}[(M_T^*)^p]$ are infinite. The theorem says that these quantities are either both finite (then the stated bounds hold) or both infinite.

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