Math 562 Fall 2020

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Outline

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1. General Info
2. 5.2 Ito's Formula
3. 5.3 A Few Consequences of Ito's Formula

I posted HW4 in my homepage. HW4 is due 10/16 at noon. I also set up HW4 in the course Moodle page.

Outline

Theorem 5.10 (Ito's formula)

Let $X^1, \ldots, X^p$ be p continuous semimartingales, and let F be a twice continuously differentiable real-valued function on $\mathbb{R}^p$. Then, for every $t \geq 0$,

$$F(X_t^1, \ldots, X_t^p) = F(X_0^1, \ldots, X_0^p) + \sum_{j=1}^{p} \int_0^t \frac{\partial F}{\partial x^j}(X_t^1, \ldots, X_t^p) dX_s^j + \frac{1}{2} \sum_{j,k=1}^{p} \int_0^t \frac{\partial^2 F}{\partial x^j x^k}(X_t^1, \ldots, X_t^p) d\langle X^j, X^k \rangle_s.$$

Last time, we proved the theorem in the case $p = 1$. This time we deal with the general case.

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Last time, we proved the theorem in the case $p = 1$. This time we deal with the general case.

Proof of Theorem 5.10 (cont)

In the general case, the Taylor-Lagrange formula, applied for every $n \geq 1$ and every $j \in \{0, \ldots, p_n - 1\}$ to the function

$$[0,1] \ni \theta \mapsto F(X^1_{t^n_j} + \theta(X^1_{t^n_{j+1}} - X^1_{t^n_j}), \ldots, X^p_{t^n_j} + \theta(X^p_{t^n_{j+1}} - X^p_{t^n_j}))$$

gives

$$F(X^1_{t^n_{j+1}}, \cdots, X^p_{t^n_{j+1}}) - F(X^1_{t^n_j}, \cdots, X^p_{t^n_j}) = \sum_{k=1}^{p} \frac{\partial F}{\partial x^k}(X^1_{t^n_j}, \cdots, X^p_{t^n_j})(X^k_{t^n_{j+1}} - X^k_{t^n_j}) + \sum_{k,l=1}^{p} \frac{f^{k,l}_{n,j}}{2}(X^k_{t^n_{j+1}} - X^k_{t^n_j})(X^l_{t^n_{j+1}} - X^l_{t^n_j})$$

where, for $k, l \in \{1, \ldots, p\}$,

$$f^{k,l}_{n,j} = \frac{\partial^2 F}{\partial x^k \partial x^l}(X_{t^n_j} + c(X_{t^n_{j+1}} - X_{t^n_j}))$$

for some $c \in [0,1]$ (here we use the notation $X_t = (X^1_t, \ldots, X^p_t)$).

Proof of Theorem 5.10 (cont)

Proposition 5.9 can again be used to handle the terms involving first derivatives. Moreover, a slight modification of the arguments of the case $p = 1$ shows that, at least along a suitable sequence of values of n, we have for every $k, l \in \{1, \dots, p\}$,

$$\lim_{n\to\infty} \sum_{j=1}^{p_n-1} f_{n,j}^{k,l} (X^k_{t^n_{j+1}} - X^k_{t^n_j})(X^l_{t^n_{j+1}} - X^l_{t^n_j})$$

$$= \int_0^t \frac{\partial^2 F}{\partial x^k \partial x^l}(X^1_s, \dots, X^p_s) d\langle X^k, X^l\rangle_s$$

in probability. This completes the proof of the theorem.

An important special case of Ito's formula is the integration by parts formula, which is obtained by taking $p = 2$ and $F(x, y) = xy$: if X and Y are two continuous semimartingales, we have

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle_t.$$

In particular, if $Y = X$,

$$X_t^2 = X_0^2 + 2 \int_0^t X_s dX_s + \langle X, X \rangle_t.$$

When $X = M$ is a continuous local martingale, we know from the definition of the quadratic variation that $M_t^2 - \langle M, M \rangle_t$ is a continuous local martingale. The previous formula shows that this continuous local martingale is

$$M_0^2 + 2 \int_0^t M_s dM_s.$$

Let B be a 1-dim $(\mathcal{F}_t)$-Brownian motion. B is a continuous local martingale (a martingale if $B_0 \in L^1$) and $\langle B, B \rangle_t = t$. In this particular case, Ito's formula reads

$$F(B_t) = F(B_0) + \int_0^t F'(B_s)dB_s + \frac{1}{2}\int_0^t F''(B_s)ds.$$

Taking $X_t^1 = t, X_t^2 = B_t$, we also get for every twice continuously differentiable function $F(t, x)$ on $\mathbb{R}_+ \times \mathbb{R}$,

$$F(t, B_t) = F(0, B_0) + \int_0^t \frac{\partial F}{\partial x}(s, B_s)dB_s + \int_0^t (\frac{\partial F}{\partial s} + \frac{\partial^2 F}{\partial x^2})(s, B_s)ds.$$

Let $B_t = (B_t^1, \ldots, B_t^d)$ be a d-dim $(\mathcal{F}_t)$-Brownian motion. Note that the components $B^1, \ldots, B^d$ are $(\mathcal{F}_t)$-Brownian motions. By Proposition 4.16, $\langle B^i, B^j \rangle_t = 0$ if $i \neq j$. Ito's formula then shows that, for every twice continuously differentiable function F on $\mathbb{R}^d$,

$$\begin{aligned} &F(B_t^1, \ldots, B_t^d) - F(B_0^1, \ldots, B_0^d) \\ &= \sum_i^d \int_0^t \frac{\partial F}{\partial x^i}(B_s^1, \ldots, B_s^d) dB_s^i + \frac{1}{2} \int_0^t \Delta F(B_s^1, \ldots, B_s^d) ds. \end{aligned}$$

The latter formula is often written in the shorter form

$$F(B_t) = F(B_0) + \int_0^t \nabla F(B_s) \cdot dB_s + \frac{1}{2} \int_0^t \Delta F(B_s) ds.$$

It frequently occurs that one needs to apply Ito's formula to a function F which is only defined (and twice continuously differentiable) on an open subset U of $\mathbb{R}^p$. In this case, we can argue in the following way. Suppose that there exists another open set V, such that $(X_0^1, \cdots, X_0^p) \in V$ a.s and $\overline{V} \subset U$. Typically V will be the set of all points whose distance from U^c is strictly greater than some $\epsilon > 0$. Define $T_v = \inf\{t \geq 0 : X_t \notin V\}$, which is a stopping time. Simple analytic arguments allow us to find a function G which is twice continuously differentiable on $\mathbb{R}^p$ and coincides with F on $\overline{V}$. We can now apply Ito's formula obtain the canonical decomposition of the semimartingale $G(X^1_{t\wedge T_V}, \ldots, X^p_{t\wedge T_V}) = F(X^1_{t\wedge T_V}, \ldots, X^p_{t\wedge T_V})$ and this decomposition only involves the first and second derivatives of F on V. If in addition we know that the process $(X_t^1, \cdots, X_t^p)$ a.s. does not exit U, we can let the open set V increase to U, and we get that Ito's formula for $F(X_t^1, \ldots, X_t^p)$ remains valid exactly in the same form as in Theorem 5.10. These considerations can be applied, for instance, to the function $F(x) = \log x$ and to a semimartingale X taking strictly positive values.

A process with values in the complex plane $\mathbb{C}$ is called a complex continuous local martingale if both its real part and its imaginary part are continuous local martingales.

Proposition 5.11

Let M be a continuous local martingale and, for every $\lambda \in \mathbb{C}$, let

$$\mathcal{E}(\lambda M)_t = \exp\left(\lambda M_t - \frac{\lambda^2}{2}\langle M, M\rangle_t\right).$$

The process $\mathcal{E}(\lambda M)$ s a complex continuous local martingale, which can be written in the form

$$\mathcal{E}(\lambda M)_t = e^{\lambda M_0} + \lambda \int_0^t \mathcal{E}(\lambda M)_s dM_s.$$

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Proof of Proposition 5.11

If $F(r,x)$ is a twice continuously differentiable function on $\mathbb{R}^2$, Ito's formula gives

$$F(\langle M,M\rangle_t, M_t) = F(0,M_0) + \int_0^t \frac{\partial F}{\partial x}(\langle M,M\rangle_s, M_s)dM_s + \int_0^t \left(\frac{\partial F}{\partial r} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\right)(\langle M,M\rangle_s, M_s)d\langle M,M\rangle_s.$$

Hence, $F(\langle M,M\rangle_t, M_t)$ is a continuous local martingale as soon as F satisfies the equation

$$\frac{\partial F}{\partial r} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} = 0.$$

This equation holds for $F(r,x) = \exp(\lambda x - \frac{\lambda^2}{2}r)$. Moreover, for this choice of F we have $\frac{\partial F}{\partial x} = \lambda F$, which leads to the formula of the statement.

Outline

Theorem 5.12

Let $X = (X^1, \ldots, X^d)$ be a continuous $(\mathcal{F}_t)$-adapted process. The following are equivalent:

(i) X is a d-dim $(\mathcal{F}_t)$-Brownian motion.

(ii) The processes $X^1, \ldots, X^d$ are continuous local martingales, and $\langle X^i, X^j \rangle_t = \delta_{ij} t$ for every $i, j \in \{1, \ldots, d\}$.

In particular, a continuous local martingale M is an $(\mathcal{F}_t)$-Brownian motion if and only if $\langle M, M \rangle_t = t$ for every $t \geq 0$, or equivalently if and only if $M_t^2 - t$ is a continuous local martingale.

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(i)$\Rightarrow$(ii) is known. We only need to show (ii)$\Rightarrow$(i). So we assume (ii) holds. Let $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$. Then $\xi \cdot X_t = \sum_{j=1}^d \xi_j X_t^j$ is a cont

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Proof of Theorem 5.12 (cont)

local martingale with quadratic variation

$$\sum_{j=1}^{d}\sum_{k=1}^{d}\xi_j\xi_k\langle X^j, X^k\rangle_t = |\xi|^2 t.$$

By Proposition 5.11, $\exp(i\xi \cdot X_t + \frac{1}{2}|\xi|^2 t)$ is a complex continuous local martingale. This complex continuous local martingale is bounded on every interval $[0, a]$, $a > 0$, and is therefore a (true) martingale, in the sense that its real and imaginary parts are both martingales. Hence, for every $0 \leq s < t$,

$$\mathbb{E}\left[\exp(i\xi \cdot X_t + \frac{1}{2}|\xi|^2 t)|\mathcal{F}_s\right] = \exp(i\xi \cdot X_s + \frac{1}{2}|\xi|^2 s).$$

and thus

$$\mathbb{E}\left[\exp(i\xi \cdot (X_t - X_s)|\mathcal{F}_s\right] = \exp(-\frac{1}{2}|\xi|^2 (t-s)).$$

Proof of Theorem 5.12 (cont)

It follows that, for every $A \in \mathcal{F}_s$,

$$\mathbb{E}\left[1_A \exp(i\xi \cdot (X_t - X_s)\right] = \mathbb{P}(A) \exp(-\frac{1}{2}|\xi|^2(t-s)).$$

Taking $A = \Omega$, we get that $X_t - X_s$ is a centered Gaussian vector with covariance matrix $(t-s)Id$ (in particular, the components $X_t^j - X_s^j$, $1 \leq j \leq d$, are independent). Furthermore, fix $A \in \mathcal{F}_s$ with $\mathbb{P}(A) > 0$, and write $\mathbb{P}_A$ for the conditional probability $\mathbb{P}_A(\cdot) = \mathbb{P}(\cdot|A)$. We also obtain that

$$\mathbb{E}_{\mathbb{P}_A}\left[\exp(i\xi \cdot (X_t - X_s)\right] = \exp(-\frac{1}{2}|\xi|^2(t-s))$$

which means that the law of $X_t - X_s$ under $\mathbb{P}_A$ is the same as its law under $\mathbb{P}$. Therefore, for any non-negative measurable function f on $\mathbb{R}^d$, we have

Proof of Theorem 5.12 (cont)

$$\mathbb{E}_{\mathbb{P}_A}\left[f(X_t - X_s)\right] = \mathbb{E}\left[f(X_t - X_s)\right]$$

or equivalently

$$\mathbb{E}\left[1_A f(X_t - X_s)\right] = \mathbb{P}(A)\mathbb{E}\left[f(X_t - X_s)\right].$$

This holds for any $A \in \mathcal{F}_s$ (when $\mathbb{P}(A) = 0$ this equality is trivial), and thus $X_t - X_s$ is independent of $\mathcal{F}_s$.
It follows that, if $0 = t_0 < t_s < \cdots < t_p$, the vectors $X_{t_1} - X_{t_0}, \cdots, X_{t_p} - X_{t_{p-1}}$ are independent. Since the components of each of these vectors are independent random variables, we obtain that all variables $X^j_{t_k} - X^j_{t_{k-1}}$, $1 \leq j \leq d$, $1 \leq k \leq p$, are independent, and $X^j_{t_k} - X^j_{t_{k-1}}$ is distributed according to $\mathcal{N}(0, t_k - t_{k-1})$. This implies that $X_t - X_0$ is a d-dimensional Brownian motion started from 0.

Proof of Theorem 5.12 (cont)

Since we also know that $X - X_0$ is independent of X_0, we get that X is a d-dim Brownian motion. Finally, X is adapted and has independent increments with respect to the filtration $(\mathcal{F}_t)$, so that X is a d-dim $(\mathcal{F}_t)$-Brownian motion.

The next theorem shows that any continuous local martingale M can be written as a "time-changed" Brownian motion. It follows that the sample paths of M are Brownian sample paths run at a different (varying) speed, and certain almost sure properties of sample paths of M can be deduced from the corresponding properties of Brownian sample paths. For instance, under the condition $\langle M, M\rangle_\infty = \infty$, the sample paths of M must oscillate between $-\infty$ and ∞ as $t \to \infty$.

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Theorem 5.13 (Dambis-Dubins-Schwarz)

Let M be a continuous local martingale such that $\langle M, M\rangle_\infty = \infty$ a.s. There exists a Brownian motion $(\beta_s)_{s\geq 0}$ such that

$$a.s. \quad \forall t \geq 0, \quad M_t = \beta_{\langle M,M\rangle_t}.$$

Remarks

(i) One can remove the assumption $\langle M, M\rangle_\infty = \infty$, at the cost of enlarging the underlying probability space.

(ii) The Brownian motion β is not adapted with respect to the filtration $(\mathcal{F}_t)$, but with respect to a "time-changed" filtration, as the proof will show.

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