Outline

1. General Info
2. 5.2 Ito’s Formula
3. 5.3 A Few Consequences of Ito’s Formula
I posted HW4 in my homepage. HW4 is due 10/16 at noon. I also set up HW4 in the course Moodle page.
Outline

1. General Info
2. 5.2 Ito’s Formula
3. 5.3 A Few Consequences of Ito’s Formula
**Theorem 5.10 (Ito’s formula)**

Let \( X^1, \ldots, X^p \) be \( p \) continuous semimartingales, and let \( F \) be a twice continuously differentiable real-valued function on \( \mathbb{R}^p \). Then, for every \( t \geq 0 \),

\[
F(X^1_t, \ldots, X^p_t) = F(X^1_0, \ldots, X^p_0) + \sum_{j=1}^{p} \int_0^t \frac{\partial F}{\partial x^j}(X^1_t, \ldots, X^p_t) \, dX^j_s
+ \frac{1}{2} \sum_{j,k=1}^{p} \int_0^t \frac{\partial^2 F}{\partial x^j x^k}(X^1_t, \ldots, X^p_t) \, d\langle X^j, X^k \rangle_s.
\]

Last time, we proved the theorem in the case \( p = 1 \). This time we deal with the general case.
Theorem 5.10 (Ito’s formula)

Let $X_1, \ldots, X^p$ be $p$ continuous semimartingales, and let $F$ be a twice continuously differentiable real-valued function on $\mathbb{R}^p$. Then, for every $t \geq 0$,

$$F(X_{t}^{1}, \ldots, X_{t}^{p}) = F(X_{0}^{1}, \ldots, X_{0}^{p}) + \sum_{j=1}^{p} \int_{0}^{t} \frac{\partial F}{\partial x^j}(X_{s}^{1}, \ldots, X_{s}^{p})dX_{s}^{j}$$

$$+ \frac{1}{2} \sum_{j,k=1}^{p} \int_{0}^{t} \frac{\partial^2 F}{\partial x^j x^k}(X_{s}^{1}, \ldots, X_{s}^{p})d\langle X_{s}^{j}, X_{s}^{k} \rangle.$$ 

Last time, we proved the theorem in the case $p = 1$. This time we deal with the general case.
Proof of Theorem 5.10 (cont)

In the general case, the Taylor-Lagrange formula, applied for every \( n \geq 1 \) and every \( j \in \{0, \ldots, p_n - 1\} \) to the function

\[
[0, 1] \ni \theta \mapsto F(X_{t_{j+1}^n}^1 + \theta(X_{t_{j+1}^n}^1 - X_{t_j^n}^1), \ldots, X_{t_{j+1}^n}^p + \theta(X_{t_{j+1}^n}^p - X_{t_j^n}^p))
\]

gives

\[
F(X_{t_{j+1}^n}^1, \ldots, X_{t_{j+1}^n}^p) - F(X_{t_j^n}^1, \ldots, X_{t_j^n}^p) = \sum_{k=1}^p \frac{\partial F}{\partial x_k} (X_{t_j^n}^1, \ldots, X_{t_j^n}^p)(X_{t_{j+1}^n}^k - X_{t_j^n}^k)
\]

\[
+ \sum_{k,l=1}^p \frac{f_{n,j}^{k,l}}{2} (X_{t_{j+1}^n}^k - X_{t_j^n}^k)(X_{t_{j+1}^n}^l - X_{t_j^n}^l)
\]

where, for \( k, l \in \{1, \ldots, p\} \),

\[
f_{n,j}^{k,l} = \frac{\partial^2 F}{\partial x_k \partial x_l} (X_{t_j^n} + c(X_{t_{j+1}^n} - X_{t_j^n}))
\]

for some \( c \in [0, 1] \) (here we use the notation \( X_t = (X_t^1, \ldots, X_t^p) \)).
Proof of Theorem 5.10 (cont)

Proposition 5.9 can again be used to handle the terms involving first derivatives. Moreover, a slight modification of the arguments of the case $p = 1$ shows that, at least along a suitable sequence of values of $n$, we have for every $k, l \in \{1, \ldots, p\}$,

$$\lim_{n \to \infty} \sum_{j=1}^{p_n-1} f_{n,j}^{k,l} (X_{t_j}^k - X_{t_{j+1}}^k)(X_{t_j}^l - X_{t_{j+1}}^l)$$

$$= \int_0^t \frac{\partial^2 F}{\partial x^k \partial x^l} (X_s^1, \ldots, X_s^p) d\langle X^k, X^l \rangle_s$$

in probability. This completes the proof of the theorem.
An important special case of Ito's formula is the integration by parts formula, which is obtained by taking $p = 2$ and $F(x, y) = xy$: if $X$ and $Y$ are two continuous semimartingales, we have

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle_t.$$ 

In particular, if $Y = X$,

$$X_t^2 = X_0^2 + 2 \int_0^t X_s dX_s + \langle X, X \rangle_t.$$ 

When $X = M$ is a continuous local martingale, we know from the definition of the quadratic variation that $M_t^2 - \langle M, M \rangle_t$ is a continuous local martingale. The previous formula shows that this continuous local martingale is

$$M_0^2 + 2 \int_0^t M_s dM_s.$$
Let $B$ be a 1-dim $(\mathcal{F}_t)$-Brownian motion. $B$ is a continuous local martingale (a martingale if $B_0 \in L^1$) and $\langle B, B \rangle_t = t$. In this particular case, Ito’s formula reads

$$F(B_t) = F(B_0) + \int_0^t F'(B_s)dB_s + \frac{1}{2} \int_0^t F''(B_s)ds.$$

Taking $X^1_t = t$, $X^2_t = B_t$, we also get for every twice continuously differentiable function $F(t, x)$ on $\mathbb{R}_+ \times \mathbb{R}$,

$$F(t, B_t) = F(0, B_0) + \int_0^t \frac{\partial F}{\partial x}(s, B_s)dB_s + \int_0^t \left( \frac{\partial F}{\partial s} + \frac{\partial^2 F}{\partial x^2} \right)(s, B_s)ds.$$
Let \( B_t = (B^1_t, \ldots, B^d_t) \) be a \( d \)-dim \((\mathcal{F}_t)\)-Brownian motion. Note that the components \( B^1, \ldots, B^d \) are \((\mathcal{F}_t)\)-Brownian motions. By Proposition 4.16, \( \langle B^i, B^j \rangle_t = 0 \) if \( i \neq j \). Ito’s formula then shows that, for every twice continuously differentiable function \( F \) on \( \mathbb{R}^d \),

\[
F(B^1_t, \ldots, B^d_t) - F(B^1_0, \ldots, B^d_0) = \sum_{i}^{d} \int_{0}^{t} \frac{\partial F}{\partial x^i}(B^1_s, \ldots, B^d_s)dB^i_s + \frac{1}{2} \int_{0}^{t} \Delta F(B^1_s, \ldots, B^d_s)ds.
\]

The latter formula is often written in the shorter form

\[
F(B_t) = F(B_0) + \int_{0}^{t} \nabla F(B_s) \cdot dB_s + \frac{1}{2} \int_{0}^{t} \Delta F(B_s)ds.
\]
It frequently occurs that one needs to apply Ito’s formula to a function $F$ which is only defined (and twice continuously differentiable) on an open subset $U$ of $\mathbb{R}^p$. In this case, we can argue in the following way. Suppose that there exists another open set $V$, such that $(X_0^1, \ldots, X_0^p) \in V \text{ a.s and } \overline{V} \subset U$. Typically $V$ will be the set of all points whose distance from $U^c$ is strictly greater than some $\epsilon > 0$. Define $T_V = \inf\{ t \geq 0 : X_t \not\in V \}$, which is a stopping time. Simple analytic arguments allow us to find a function $G$ which is twice continuously differentiable on $\mathbb{R}^p$ and coincides with $F$ on $\overline{V}$. We can now apply Ito’s formula obtain the canonical decomposition of the semimartingale $G(X_{t\wedge T_V}^1, \ldots, X_{t\wedge T_V}^p) = F(X_{t\wedge T_V}^1, \ldots, X_{t\wedge T_V}^p)$ and this decomposition only involves the first and second derivatives of $F$ on $V$. If in addition we know that the process $(X_t^1, \ldots, X_t^p)$ a.s. does not exit $U$, we can let the open set $V$ increase to $U$, and we get that Ito’s formula for $F(X_t^1, \ldots, X_t^p)$ remains valid exactly in the same form as in Theorem 5.10. These considerations can be applied, for instance, to the function $F(x) = \log x$ and to a semimartingale $X$ taking strictly positive values.
A process with values in the complex plane $\mathbb{C}$ is called a complex continuous local martingale if both its real part and its imaginary part are continuous local martingales.

**Proposition 5.11**

Let $M$ be a continuous local martingale and, for every $\lambda \in \mathbb{C}$, let

$$
\mathcal{E}(\lambda M)_t = \exp \left( \lambda M_t - \frac{\lambda^2}{2} \langle M, M \rangle_t \right).
$$

The process $\mathcal{E}(\lambda M)$ is a complex continuous local martingale, which can be written in the form

$$
\mathcal{E}(\lambda M)_t = e^{\lambda M_0} + \lambda \int_0^t \mathcal{E}(\lambda M)_s dM_s.
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\[
\mathcal{E}(\lambda M)_t = e^{\lambda M_0} + \lambda \int_0^t \mathcal{E}(\lambda M)_s dM_s.
\]
Proof of Proposition 5.11

If \( F(r, x) \) is a twice continuously differentiable function on \( \mathbb{R}^2 \), Ito’s formula gives

\[
F(\langle M, M \rangle_t, M_t) = F(0, M_0) + \int_0^t \frac{\partial F}{\partial x} (\langle M, M \rangle_s, M_s) dM_s \\
+ \int_0^t \left( \frac{\partial F}{\partial r} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \right) (\langle M, M \rangle_s, M_s) d\langle M, M \rangle_s.
\]

Hence, \( F(\langle M, M \rangle_t, M_t) \) is a continuous local martingale as soon as \( F \) satisfies the equation

\[
\frac{\partial F}{\partial r} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} = 0.
\]

This equation holds for \( F(r, x) = \exp(\lambda x - \frac{\lambda^2}{2} r) \). Moreover, for this choice of \( F \) we have \( \frac{\partial F}{\partial x} = \lambda F \), which leads to the formula of the statement.
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Theorem 5.12

Let $X = (X^1, \ldots, X^d)$ be a continuous $(\mathcal{F}_t)$-adapted process. The following are equivalent:

(i) $X$ is a $d$-dim $(\mathcal{F}_t)$-Brownian motion.

(ii) The processes $X^1, \ldots, X^d$ are continuous local martingales, and $\langle X^i, X^j \rangle_t = \delta_{ij} t$ for every $i, j \in \{1, \ldots, d\}$.

In particular, a continuous local martingale $M$ is an $(\mathcal{F}_t)$-Brownian motion if and only if $\langle M, M \rangle_t = t$ for every $t \geq 0$, or equivalently if and only if $M_t^2 - t$ is a continuous local martingale.

Proof of Theorem 5.12

$(i) \implies (ii)$ is known. We only need to show $(ii) \implies (i)$. So we assume $(ii)$ holds. Let $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$. Then $\xi \cdot X_t = \sum_{j=1}^d \xi_j X^j_t$ is a cont
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Proof of Theorem 5.12 (cont)

local martingale with quadratic variation

\[ \sum_{j=1}^{d} \sum_{k=1}^{d} \xi_j \xi_k \langle X^j, X^k \rangle_t = |\xi|^2 t. \]

By Proposition 5.11, \( \exp(i \xi \cdot X_t + \frac{1}{2} |\xi|^2 t) \) is a complex continuous local martingale. This complex continuous local martingale is bounded on every interval \([0, a], a > 0\), and is therefore a (true) martingale, in the sense that its real and imaginary parts are both martingales. Hence, for every \( 0 \leq s < t \),

\[ \mathbb{E} \left[ \exp(i \xi \cdot X_t + \frac{1}{2} |\xi|^2 t) | F_s \right] = \exp(i \xi \cdot X_s + \frac{1}{2} |\xi|^2 s). \]

and thus

\[ \mathbb{E} \left[ \exp(i \xi \cdot (X_t - X_s) | F_s \right] = \exp(-\frac{1}{2} |\xi|^2 (t - s)). \]
Proof of Theorem 5.12 (cont)

It follows that, for every $A \in \mathcal{F}_s$,

$$
\mathbb{E} [1_A \exp(i\xi \cdot (X_t - X_s))] = \mathbb{P}(A) \exp(-\frac{1}{2}|\xi|^2(t - s)).
$$

Taking $A = \Omega$, we get that $X_t - X_s$ is a centered Gaussian vector with covariance matrix $(t - s)\text{id}$ (in particular, the components $X_t^j - X_s^j$, $1 \leq j \leq d$, are independent). Furthermore, fix $A \in \mathcal{F}_s$ with $\mathbb{P}(A) > 0$, and write $\mathbb{P}_A$ for the conditional probability $\mathbb{P}_A(\cdot) = \mathbb{P}(\cdot | A)$. We also obtain that

$$
\mathbb{E}_{\mathbb{P}_A} [\exp(i\xi \cdot (X_t - X_s))] = \exp(-\frac{1}{2}|\xi|^2(t - s))
$$

which means that the law of $X_t - X_s$ under $\mathbb{P}_A$ is the same as its law under $\mathbb{P}$. Therefore, for any non-negative measurable function $f$ on $\mathbb{R}^d$, we have
Proof of Theorem 5.12 (cont)

\[ \mathbb{E}_{\mathbb{P}_A} [f(X_t - X_s)] = \mathbb{E} [f(X_t - X_s)] \]

or equivalently

\[ \mathbb{E} [1_A f(X_t - X_s)] = \mathbb{P}(A) \mathbb{E} [f(X_t - X_s)]. \]

This holds for any \( A \in \mathcal{F}_s \) (when \( \mathbb{P}(A) = 0 \) this equality is trivial), and thus \( X_t - X_s \) is independent of \( \mathcal{F}_s \).

It follows that, if \( 0 = t_0 < t_s < \cdots < t_p \), the vectors \( X_{t_1} - X_{t_0}, \cdots, X_{t_p} - X_{t_{p-1}} \) are independent. Since the components of each of these vectors are independent random variables, we obtain that all variables \( X_{t_k}^j - X_{t_{k-1}}^j, 1 \leq j \leq d, 1 \leq k \leq p \), are independent, and \( X_{t_k}^j - X_{t_{k-1}}^j \) is distributed according to \( \mathcal{N}(0, t_k - t_{k-1}) \). This implies that \( X_t - X_0 \) is a \( d \)-dimensional Brownian motion started from 0.
Proof of Theorem 5.12 (cont)

Since we also know that $X - X_0$ is independent of $X_0$, we get that $X$ is a $d$-dim Brownian motion. Finally, $X$ is adapted and has independent increments with respect to the filtration $(\mathcal{F}_t)$, so that $X$ is a $d$-dim $(\mathcal{F}_t)$-Brownian motion.

The next theorem shows that any continuous local martingale $M$ can be written as a “time-changed” Brownian motion. It follows that the sample paths of $M$ are Brownian sample paths run at a different (varying) speed, and certain almost sure properties of sample paths of $M$ can be deduced from the corresponding properties of Brownian sample paths. For instance, under the condition $\langle M, M \rangle_\infty = \infty$, the sample paths of $M$ must oscillate between $-\infty$ and $\infty$ as $t \to \infty$. 
Proof of Theorem 5.12 (cont)

Since we also know that \( X - X_0 \) is independent of \( X_0 \), we get that \( X \) is a \( d \)-dim Brownian motion. Finally, \( X \) is adapted and has independent increments with respect to the filtration \( (\mathcal{F}_t) \), so that \( X \) is a \( d \)-dim \( (\mathcal{F}_t) \)-Brownian motion.

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Theorem 5.13 (Dambis-Dubins-Schwarz)

Let $M$ be a continuous local martingale such that $\langle M, M \rangle_\infty = \infty$ a.s. There exists a Brownian motion $(\beta_s)_{s \geq 0}$ such that

$$a.s. \quad \forall t \geq 0, \quad M_t = \beta_{\langle M, M \rangle_t}.$$

Remarks

(i) One can remove the assumption $\langle M, M \rangle_\infty = \infty$, at the cost of enlarging the underlying probability space.

(ii) The Brownian motion $\beta$ is not adapted with respect to the filtration $(\mathcal{F}_t)$, but with respect to a “time-changed” filtration, as the proof will show.
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