Math 562 Fall 2020

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Outline

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I posted HW4 in my homepage. HW4 is due 10/16 at noon. I also set up HW4 in the course Moodle page.

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Definition 5.7

Let X be a continuous semimartingale and let $X = M + V$ be its canonical decomposition. If H is a locally bounded progressive process, the stochastic integral $H \cdot X$ is the continuous semimartingale with canonical decomposition

$$H \cdot X = H \cdot M + H \cdot V$$

and we write

$$(H \cdot X)_t = \int_0^t H_s dX_s.$$

Properties

(i) The mapping $(H, X) \mapsto H \cdot X$ is bilinear.

(ii) $H \cdot (K \cdot X) = (HK) \cdot X$ if H and K are progressive and locally bounded.

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Properties (cont)

(iii) For every stopping time T, $(H \cdot X)^T = H1_{[0,T]} \cdot X = H \cdot X^T$.

(iv) If X is a continuous local martingale, resp. if X is a finite variation process, then the same holds for $H \cdot X$.

(v) If H is of the form $H_s(\omega) = \sum_{j=0}^{p-1} H_{(j)}(\omega) 1_{(t_j, t_{j+1}]}(s)$, where $0 = t_0 < t_1 < \cdots < t_p$, and, for every $j = 0, \ldots, p-1$, $H_{(j)}$ is $\mathcal{F}_{t_j}$-measurable, then

$$(H \cdot X)_t = \sum_{j=0}^{p-1} H_{(j)}(X_{t_{j+1} \wedge t} - X_{t_j \wedge t}).$$

Property (ii) can be restated as, if $Y_t = \int_0^t K_s dX_s$ then

$$\int_0^t H_s dY_s = \int_0^t H_s K_s dX_s.$$

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Properties (i)–(iv) easily follow from the results obtained when X is a continuous local martingale, resp. a finite variation process. As for property (v), we first note that it is enough to consider the case where $X = M$ is a continuous local martingale with $M_0 = 0$, and by stopping M at suitable stopping times, we can even assume that M is in $\mathbb{H}^2$. There is a minor difficulty coming from the fact that the variables $H_{(j)}$ are not assumed to be bounded (and therefore we cannot directly use the construction of the integral of elementary processes). To circumvent this difficulty, we set, for every $n \geq 1$,

$$T_n = \inf\{t \geq 0 : |H_t| \geq n\} = \inf\{t_j : |H_{(j)}| \geq n\}.$$

It is easy to verify that T_n is a stopping time, and we have $T_n \uparrow \infty$ as $n \uparrow \infty$.

Furthermore, we have for every n,

$$H_s 1_{[0,T_n]}(s) = \sum_{j=0}^{p-1} H^n_{(j)} 1_{(t_j,t_{j+1}]}(s)$$

where the random variables $H^n_{(j)} = H_{(j)} 1_{\{T_n > t_j\}}$ satisfy the same properties as the $H_{(j)}$'s and additionally are bounded by n. Hence $H 1_{[0,T_n]}$ is an elementary process, and by the very definition of the stochastic integral with respect to a martingale of $\mathbb{H}^2$, we have

$$(H \cdot M)_{t \wedge T_n} = (H 1_{[0,T_n]} \cdot M)_t = \sum_{j=0}^{p-1} H^n_{(j)} (M_{t_{j+1} \wedge t} - M_{t_j \wedge t})$$

The desired result now follows by letting n tend to infinity.

Proposition 5.8

Let $X = M + V$ be the canonical decomposition of a continuous semimartingale X, and let $t > 0$. Let $(H^n)_{n \geq 1}$ and H be locally bounded progressive processes, and let K be a non-negative progressive process. Assume that the following properties hold a.s.:

(i) $H^n_s \to H_s$ as $n \to \infty$, for every $s \in [0, t]$;

(ii) $|H^n_s| \leq K_s$, for every $n \geq 1$ and $s \in [0, t]$;

(iii) $\int_0^t K_s^2 d\langle M, M\rangle_s < \infty$ and $\int_0^t K_s|dV_s| < \infty$.

Than, as $n \to \infty$,

$$\int_0^t H^n_s dX_s \to \int_0^t H_s dX_s$$

in probability.

Remarks

(a) Condition (iii) holds automatically if K is locally bounded.

(b) Instead of assuming that (i) and (ii) hold for every $s \in [0;t]$ (a.s.), it is enough to assume that these conditions hold for $d\langle M,M\rangle_s$-a.e, $s \in [0,t]$ and for $|dV_s|$-a.e. $s \in [0,t]$, a.s. This will be clear from the proof.

Proof of Proposition 5.8

The a.s. convergence

$$\int_0^t H_s^n dV_s \to \int_0^t H_s dV_s$$

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Proof of Proposition 5.8 (cont)

For every $p \geq 1$, define

$$T_p = \inf\{r \in [0,t] : \int_0^r K_s^2 d\langle M, M\rangle_s \geq n\} \wedge t.$$

Note that $T_p = t$ for all large enough p, a.s., by assumption (iii). Then

$$\mathbb{E}\left[\left(\int_0^{T_p} H_s^n dM_s - \int_0^{T_p} H_s dM_s\right)^2\right]$$
$$\leq \mathbb{E}\left[\int_0^{T_p} (H_s^n - H_s)^2 d\langle M, M\rangle_s\right]$$

which tends to 0 as $n \to \infty$, by dominated convergence, using assumptions (i) and (ii) and the fact that $\int_0^{T_p} K_s^2 d\langle M, M\rangle_s \leq p$. Since $\mathbb{P}(T_p = t) \to 1$ as $p \to \infty$, the desired result follows.

Proposition 5.9

Let X be a continuous semimartingale, and let H be an adapted process with continuous sample paths. Then, for every $t > 0$, for every sequence $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ of partitions of $[0, t]$ with mesh tending to 0,

$$\lim_{n\to\infty} \sum_{j=0}^{p_n-1} H_{t_j^n}(X_{t_{j+1}^n} - X_{t_j^n}) \to \int_0^t H_s dX_s$$

in probability.

Proof of Proposition 5.9

For every $n \geq 1$, define a process H_s^n by

$$H_s^n = \begin{cases} H_{t_j^n}, & \text{if } t_j^n < s \leq t_{j+1}^n, j = 0, \ldots, p_n - 1 \\ H_0, & \text{if } s = 0 \\ 0, & \text{if } s > t. \end{cases}$$

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Proof of Proposition 5.9 (cont)

Note that H^n is progressive. We then observe that all assumptions of Proposition 5.8 hold if we take

$$K_s = \max_{0 \leq r \leq s} |H_r|$$

which is a locally bounded progressive process. Hence, we conclude that

$$\int_0^t H_s^n dX_s \to \int_0^t H_s dX_s$$

in probability. This gives the desired result since

$$\int_0^t H_s^n dX_s = \sum_{j=0}^{p_n - 1} H_{t_j^n} (X_{t_{j+1}^n} - X_{t_j^n}).$$

Remark

The preceding proposition can be viewed as a generalization of the corresponding result for Lebesgue-Stieltjes integrals to stochastic integrals. However, in contrast with that lemma, it is essential in Proposition 5.9 to evaluate H at the left end of the interval $(t_j^n, t_{j+1}^n]$. The result will fail if we replace $H_{t_j^n}$ by $H_{t_{j+1}^n}$. Here is a simple counterexample. Let $H_t = X_t$ and we assume that the sequence of partitions $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ is increasing. By the proposition, we have

$$\lim_{n\to\infty} \sum_{j=0}^{p_n-1} X_{t_j^n}(X_{t_{j+1}^n} - X_{t_j^n}) \to \int_0^t X_s dX_s$$

in probability. On the other hand, writing

$$\sum_{j=0}^{p_n-1} X_{t_{j+1}^n}(X_{t_{j+1}^n} - X_{t_j^n}) = \sum_{j=0}^{p_n-1} X_{t_j^n}(X_{t_{j+1}^n} - X_{t_j^n}) + \sum_{j=0}^{p_n-1} (X_{t_{j+1}^n} - X_{t_j^n})^2$$

Remark (cont)

and using Proposition 4.21, we get

$$\lim_{n\to\infty}\sum_{j=0}^{p_n-1} X_{t_{j+1}^n}(X_{t_{j+1}^n}-X_{t_j^n}) = \int_0^t X_s dX_s + \langle X, X\rangle_t$$

in probability. The resulting limit is different from $\int_0^t X_s dX_s$ unless the martingale part of X is zero.

Outline

Ito's formula is the most important result in stochastic calculus. It shows that, if we apply a twice continuously differentiable function to a p-tuple of continuous semimartingales, the resulting process is still a continuous semimartingale, and there is an explicit formula for the canonical decomposition of this semimartingale.

Theorem 5.10 (Ito's formula)

Let $X^1, \ldots, X^p$ be p continuous semimartingales, and let F be a twice continuously differentiable real function on $\mathbb{R}^p$. Then, for every $t \geq 0$,

$$\begin{aligned} F(X_t^1, \ldots, X_t^p) = & F(X_0^1, \ldots, X_0^p) + \sum_{j=1}^{p} \int_0^t \frac{\partial F}{\partial x^j}(X_t^1, \ldots, X_t^p) dX_s^j \\ & + \frac{1}{2} \sum_{j,k=1}^{p} \int_0^t \frac{\partial^2 F}{\partial x^j x^k}(X_t^1, \ldots, X_t^p) d\langle X^j, X^k \rangle_s. \end{aligned}$$

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Proof of Theorem 5.10

We first deal with the $p = 1$ and we write $X = X^1$ for simplicity. Fix $t > 0$ and consider an increasing sequence $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ of partitions of $[0, t]$ with mesh tending to zero. Then, for every n,

$$F(X_t) = F(X_0) + \sum_{j=0}^{p_n - 1} (F(X_{t_{j+1}^n}) - F(X_{t_j^n})).$$

For every $j = 0, \ldots, p_n - 1$, we apply the Taylor-Lagrange formula to the function $[0, 1] \ni \theta \mapsto F(X_{t_j^n} + \theta(X_{t_{j+1}^n} - X_{t_j^n}))$, we get

$$F(X_{t_{j+1}^n}) - F(X_{t_j^n}) = F'(X_{t_j^n})(X_{t_{j+1}^n} - X_{t_j^n}) + \frac{1}{2} f_{n,j}(X_{t_{j+1}^n} - X_{t_j^n})^2,$$

where $f_{n,j}$ can be written as $F''(X_{t_j^n} + c(X_{t_{j+1}^n} - X_{t_j^n}))$ for some $c \in [0, 1]$.

Proof of Theorem 5.10 (cont)

By Proposition 5.9 with $H_s = F'(X_s)$, we have

$$\lim_{n\to\infty}\sum_{j=0}^{p_n-1} F'(X_{t_j^n})(X_{t_{j+1}^n} - X_{t_j^n}) = \int_0^t F'(X_s)dX_s$$

in probability. To complete the proof of the case $p = 1$ of the theorem, it is therefore enough to verify that

$$\lim_{n\to\infty}\sum_{j=0}^{p_n-1} f_{n,j}(X_{t_{j+1}^n} - X_{t_j^n})^2 = \int_0^t F''(X_s)d\langle X, X\rangle_s \qquad (1)$$

in probability. We observe that

$$\sup_{0\le j\le p_n-1} |f_{n,j} - F''(X_{t_j^n})| \le \sup_{0\le j\le p_n-1}\left(\sup_{x\in[X_{t_j^n}\wedge X_{t_{j+1}^n}, X_{t_j^n}\vee X_{t_{j+1}^n}]} |F''(x) - F''(X_{t_j^n})|\right)$$

Proof of Theorem 5.10 (cont)

The right-hand side of the preceding display tends to 0 a.s. as $n \to \infty$, as a simple consequence of the uniform continuity of F'' (and of the sample paths of X) over a compact interval.

Since $\sum_{j=1}^{p_n-1}(X_{t^n_{j+1}} - X_{t^n_j})^2$ converges in probability, it follows from the last display that

$$\left| \sum_{j=0}^{p_n-1} f_{n,j}(X_{t^n_{j+1}} - X_{t^n_j})^2 - \sum_{j=0}^{p_n-1} F''(X_{t^n_j})(X_{t^n_{j+1}} - X_{t^n_j})^2 \right| = 0$$

in probability. So the convergence (1) will follow if we can verify that

$$\lim_{n\to\infty} \sum_{j=0}^{p_n-1} F''(X_{t^n_j})(X_{t^n_{j+1}} - X_{t^n_j})^2 = \int_0^t F''(X_s)d\langle X, X\rangle_s \qquad (2)$$

in probability. In fact, we will show that (2) holds a.s. along a suitable sequence of values of n (this suffices for our needs, because we can replace the initial sequence of partitions by a subsequence).

Proof of Theorem 5.10 (cont)

To this end, we note that

$$\sum_{j=0}^{p_n-1} F''(X_{t_j^n})(X_{t_{j+1}^n} - X_{t_j^n})^2 = \int_{[0,t]} F''(X_{t_j^n})\mu_n(ds)$$

where μ_n is the random measure on $[0, t]$ defined by

$$\mu_n(dr) := \sum_{j=0}^{p_n-1} (X_{t_{j+1}^n} - X_{t_j^n})^2 \delta_{t_j^n}(dr)$$

Let $D = \{t_j^n : n \geq 1, 0 \leq j \leq p_n\} \subset [0, t]$. As a consequence of Proposition 4.21, we get for every $r \in D$,

$$\mu_n([0, r]) \to \langle X, X \rangle_r$$

in probability, as $n \to \infty$. Using a diagonal extraction, we can thus

Proof of Theorem 5.10 (cont)

find a subsequence of values of n such that, along this subsequence, we have for every $r \in D$,

$$\mu_n([0,r]) \to \langle X, X \rangle_r, \quad a.s.,$$

which implies that the sequence μ_n converges a.s. to the measure $1_{[0,t]}(r) d\langle X, X\rangle_r$, in the sense of weak convergence of finite measures. We conclude that we have

$$\int_{[0,t]} F''(X_{t_j^n})\mu_n(ds) \to \int_0^t F''(X_s) d\langle X, X\rangle_s, a.s.$$

along the chosen subsequence. This completes the proof of the case $p = 1$.