Math 562 Fall 2020

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Outline

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1. General Info
2. 5.1 The Construction of Stochastic Integrals

I posted HW4 in my homepage. HW4 is due 10/16 at noon. I also set up HW4 in the course Moodle page.

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1 General Info

2 5.1 The Construction of Stochastic Integrals

Proposition 5.5

Let $M \in \mathbb{H}^2$ and $H \in \mathbf{L}^2(M)$. If K is a progressive process, we have $KH \in L^2(M)$ if and only if $K \in L^2(H \cdot M)$. If the latter properties hold,

$$(KH) \cdot M = K \cdot (H \cdot M).$$

Proof of Proposition 5.5

By the associative property of stochastic integral with bracket, we have

$$\mathbb{E}\left[\int_0^\infty K_s^2 H_s^2 d\langle M, M\rangle_s\right] = \mathbb{E}\left[\int_0^\infty K_s^2 d\langle H \cdot M, H \cdot M\rangle_s\right],$$

which gives the first assertion. For the second one, we write for $N \in \mathbb{H}^2$,

$$\langle (KH) \cdot M, N\rangle = KH \cdot \langle M, N\rangle = K \cdot (H \cdot \langle M, N\rangle) = K \cdot \langle H \cdot M, N\rangle$$

and, by the uniqueness statement in (5.2), this implies that $(KH) \cdot M = K \cdot (H \cdot M)$.

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Moments of stochastic integrals

Let $M, N \in \mathbb{H}^2$, $H \in L^2(M)$ and $K \in L^2(N)$. Since $H \cdot M, K \cdot N \in \mathbb{H}^2$, we have, for every $t \in [0, \infty)$,

$$\mathbb{E}[\int_0^t H_s dM_s] = 0 \tag{1}$$

$$\mathbb{E}\left[\left(\int_0^t H_s dM_s\right)\left(\int_0^t K_s dN_s\right)\right] = \mathbb{E}\left[\int_0^t H_s K_s d\langle M, N\rangle_s\right] \tag{2}$$

using Proposition 4.15 (v) and the associative property of stochastic integral and bracket. In particular,

$$\mathbb{E}\left[\left(\int_0^t H_s dM_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2 d\langle M, M\rangle_s\right]. \tag{3}$$

Furthermore, since $H \cdot M$ is a (true) martingale, we also have for every $0 \leq s < t \leq \infty$,

$$\mathbb{E}[\int_0^t H_r dM_r | \mathcal{F}_s] = \int_0^s H_r dM_r \tag{4}$$

or equivalently

$$\mathbb{E}[\int_s^t H_r dM_r | \mathcal{F}_s] = 0.$$

It is important to observe that these formulas (and particularly (1) and (3)) may no longer hold for the extensions of stochastic integrals that we will now describe.

We now extend the definition of $H \cdot M$ to an arbitrary continuous local martingale. If M is a continuous local martingale, we write $L^2_{loc}(M)$ (resp. $L^2(M)$) for the collection of all progressive processes H such that

$$\int_0^t H_s^2 d\langle M, M\rangle_s < \infty, \quad \forall t \geq 0 \;\; a.s. \left(\text{ resp. } \mathbb{E}[\int_0^\infty H_s^2 d\langle M, M\rangle_s] < \infty \right).$$

Note that $L^2(M)$ (with the same identifications as in the case where $M \in \mathbb{H}^2$) can again be viewed as an "ordinary" L^2-space and thus has a Hilbert space structure.

Theorem 5.6

Let M be a continuous local martingale. For every $H \in L^2_{loc}(M)$, there exists a unique continuous local martingale with initial value 0, which is denoted by $H \cdot M$, such that, for every continuous local martingale N,

$$\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle. \tag{5}$$

If T is a stopping time, we have

$$(1_{[0,T]}H) \cdot M = (H \cdot M)^T = H \cdot M^T. \tag{6}$$

If $H \in L^2_{loc}(M)$, and K is a progressive process, we have $K \in L^2_{loc}(H \cdot M)$ if and only if $HK \in L^2_{loc}(M)$ and then

$$K \cdot (H \cdot M) = HK \cdot M. \tag{7}$$

Finally, if $M \in \mathbb{H}^2$ and $H \in L^2(M)$, the definition of $H \cdot M$ is consistent with that of Theorem 5.4.

Proof of Theorem 5.6

We may assume that $M_0 = 0$ (in the general case, we write $M = M_0 + M'$ and set $H \cdot M = H \cdot M'$, noting that $\langle M, N\rangle = \langle M', N\rangle$ for every continuous local martingale N). Also we may assume that the property $\int_0^t H_s^2 d\langle M, M\rangle_s < \infty$ for every $t \geq 0$ holds for every $\omega \in \Omega$ (on the negligible set where this fails we may replace H by 0).

F or every $n \geq 1$, define

$$T_n = \inf\{t \geq 0 : \int_0^t (1 + H_s^2) d\langle M, M\rangle_s \geq n\},$$

so that $(T_n)_{n\geq 1}$ is a sequence of stopping times that increase to ∞. Since

$$\langle M^{T_n}, M^{T_n}\rangle_t = \langle M, M\rangle_{t\wedge T_n} \leq n,$$

the stopped martingale M^{T_n} is in $\mathbb{H}^2$. Furthermore, we also have

Proof of Theorem 5.6 (cont)

$$\int_0^\infty H_s^2 d\langle M^{T_n}, M^{T_n}\rangle_s = \int_0^{T_n} H_s^2 d\langle M, M\rangle_s \leq n.$$

Hence $H \in L^2(M^{T_n})$, and the definition of $H \cdot M^{T_n}$ makes sense by Theorem 5.4. Moreover, by property (5.3), we have, if $m > n$,

$$H \cdot M^{T_n} = (H \cdot M^{T_m})^{T_n}.$$

It follows that there exists a unique process denoted by $H \cdot M$ such that, for every n,

$$(H \cdot M)^{T_n} = H \cdot M^{T_n}.$$

Clearly $H \cdot M$ has continuous sample paths and is also adapted since $(H \cdot M)_t = \lim_{n\to\infty}(H \cdot M^{T_n})_t$. Since the processes $H \cdot M^{T_n}$ are martingales in $\mathbb{H}^2$, we get that $H \cdot M$ is a continuous local martingale.

Proof of Theorem 5.6 (cont)

To verify (5), we may assume that N is a continuous local martingale such that $N_0 = 0$. For every $n \geq 1$, define $T'_n = \inf\{t \geq 0 : |N_t| \geq n\}$ and $S_n = T_n \wedge T'_n$. Then, noting that $N^{T'_n} \in \mathbb{H}^2$, we have

$$\langle H \cdot M, N\rangle^{S_n} = \langle (H \cdot M)^{T_n}, N^{T'_n}\rangle = \langle H \cdot M^{T_n}, N^{T'_n}\rangle$$
$$= H \cdot \langle M^{T_n}, N^{T'_n}\rangle = H \cdot \langle M, N\rangle^{S_n} = (H \cdot \langle M, N\rangle)^{S_n},$$

which gives the equality $\langle H \cdot M, N\rangle = H \cdot \langle M, N\rangle$, since $S_n \uparrow \infty$ as $t \uparrow \infty$. The fact that this equality (written for every continuous local martingale N) characterizes $H \cdot M$ among continuous local martingales with initial value 0 is derived from Proposition 4.12 as in the proof of Theorem 5.4.

Proof of Theorem 5.6 (cont)

(6) is then obtained by the very same arguments as in the proof of property (5.3) in Theorem 5.4 (these arguments only depended on the characteristic property (5.2) which we have just extended in (5). Similarly, the proof of (7) is analogous to the proof of Proposition 5.5.

Finally, if $M \in \mathbb{H}^2$ and $H \in L^2(M)$, the equality $\langle H \cdot M, H \cdot M \rangle = H^2 \cdot \langle M, M \rangle$ follows from (5), and implies that $H \cdot M \in \mathbb{H}^2$. Then the characteristic property (5.2) shows that the definitions of Theorems 5.4 and 5.6 are consistent.

In the setting of Theorem 5.6, we will again write

$$(H \cdot M)_t = \int_0^t H_s dM_s.$$

The formulas (5.4) and (5.5) remain valid when M and N are continuous local martingales and $H \in L^2_{loc}(M)$ and $K \in L^2_{loc}(N)$. Indeed, these formulas immediately follow from (5).

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Connection with Wiener integral

Suppose that B is an $(\mathcal{F}_t)$-Brownian motion, and $h \in L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dt)$ is a deterministic square integrable function. We can then define the Wiener integral $\int_0^t h(s)dB_s = G(h1_{[0,t]})$, where G is the Gaussian white noise associated with B. It is easy to verify that this integral coincides with the stochastic integral $(h \cdot B)_t$ which makes sense by viewing h as a (deterministic) progressive process. This is immediate when h is a simple function, and the general case follows from a density argument.

Let us now discuss the extension of the moment formulas that we stated above in the setting of Theorem 5.4. Let M be a continuous local martingale, $H \in L^2_{loc}(M)$ and $t \in [0, \infty]$. Then, under the condition

$$\mathbb{E}\left[\int_0^t H_s^2 d\langle M, M\rangle_s\right] < \infty, \tag{8}$$

we can apply Theorem 4.13 to $(H \cdot M)^t$ and get that $(H \cdot M)^t$ is a martingale in $\mathbb{H}^2$. It follows that properties (5.6) and (5.8) still hold:

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we can apply Theorem 4.13 to $(H \cdot M)^t$ and get that $(H \cdot M)^t$ is a martingale in $\mathbb{H}^2$. It follows that properties (5.6) and (5.8) still hold:

$$\mathbb{E}\left[\int_0^t H_s dM_s\right] = 0, \quad \mathbb{E}\left[\left(\int_0^t H_s dM_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2 d\langle M, M\rangle_s\right],$$

and similarly (5.9) is valid for $0 \le s \le t$. In particular (case $t = \infty$), if $H \in L^2(M)$, the continuous local martingale $H \cdot M$ is in $\mathbb{H}^2$ and its terminal value satisfies

$$\mathbb{E}\left[\left(\int_0^\infty H_s dM_s\right)^2\right] = \mathbb{E}\left[\int_0^\infty H_s^2 d\langle M, M\rangle_s\right].$$

If the condition (8) does not hold, the previous formulas may fail. However, we always have the bound

$$\mathbb{E}\left[\left(\int_0^t H_s dM_s\right)^2\right] \le \mathbb{E}\left[\int_0^t H_s^2 d\langle M, M\rangle_s\right]$$

We now extend the definition of stochastic integrals to continuous semimartingales. We say that a progressive process H is locally bounded if

$$\forall t \geq 0, \quad \sup_{s \leq t} |H_s| < \infty, \quad a.s.$$

In particular, any adapted process with continuous sample paths is a locally bounded progressive process. If H is (progressive and) locally bounded, then for every finite variation process V, we have

$$\forall t \geq 0, \quad \int_0^t |H_s||dV_s| < \infty, \quad a.s.$$

and similarly $H \in L^2_{loc}(M)$ for every continuous local martingale M.

Definition 5.7

Let X be a continuous semimartingale and let $X = M + V$ be its canonical decomposition. If H is a locally bounded progressive process, the stochastic integral $H \cdot X$ is the continuous semimartingale with canonical decomposition

$$H \cdot X = H \cdot M + H \cdot V$$

and we write

$$(H \cdot X)_t = \int_0^t H_s dX_s.$$

Properties

(i) The mapping $(H, X) \mapsto H \cdot X$ is bilinear.

(ii) $H \cdot (K \cdot X) = (HK) \cdot X$ if H and K are progressive and locally bounded.

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Properties

(i) The mapping $(H, X) \mapsto H \cdot X$ is bilinear.

(ii) $H \cdot (K \cdot X) = (HK) \cdot X$ if H and K are progressive and locally bounded.

Properties (cont)

(iii) For every stopping time T, $(H \cdot X)^T = H1_{[0,T]} \cdot X = H \cdot X^T$.

(iv) If X is a continuous local martingale, resp. if X is a finite variation process, then the same holds for $H \cdot X$.

(v) If H is of the form $H_s(\omega) = \sum_{j=0}^{p-1} H_{(j)}(\omega) 1_{(t_j, t_{j+1}]}(s)$, where $0 = t_0 < t_1 < \cdots < t_p$, and, for every $j = 0, \ldots, p-1$, $H_{(j)}$ is $\mathcal{F}_{t_j}$-measurable, then

$$(H \cdot X)_t = \sum_{j=0}^{p-1} H_{(j)}(X_{t_{j+1} \wedge t} - X_{t_j \wedge t}).$$

Property (ii) can be restated as, if $Y_t = \int_0^t K_s dX_s$ then

$$\int_0^t H_s dY_s = \int_0^t H_s K_s dX_s.$$

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(v) If H is of the form $H_s(\omega) = \sum_{j=0}^{p-1} H_{(j)}(\omega) 1_{(t_j, t_{j+1}]}(s)$, where $0 = t_0 < t_1 < \cdots < t_p$, and, for every $j = 0, \ldots, p-1$, $H_{(j)}$ is $\mathcal{F}_{t_j}$-measurable, then

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Properties (i)–(iv) easily follow from the results obtained when X is a continuous local martingale, resp. a finite variation process. As for property (v), we first note that it is enough to consider the case where $X = M$ is a continuous local martingale with $M_0 = 0$, and by stopping M at suitable stopping times, we can even assume that M is in $\mathbb{H}^2$. There is a minor difficulty coming from the fact that the variables $H_{(j)}$ are not assumed to be bounded (and therefore we cannot directly use the construction of the integral of elementary processes). To circumvent this difficulty, we set, for every $n \geq 1$,

$$T_n = \inf\{t \geq 0 : |H_t| \geq n\} = \inf\{t_j : |H_{(j)}| \geq n\}.$$

It is easy to verify that T_n is a stopping time, and we have $T_n \uparrow \infty$ as $n \uparrow \infty$.

Furthermore, we have for every n,

$$H_s 1_{[0,T_n]}(s) = \sum_{j=0}^{p-1} H^n_{(j)} 1_{(t_j, t_{j+1}]}(s)$$

where the random variables $H^n_{(j)} = H_{(j)} 1_{\{T_n > t_j\}}$ satisfy the same properties as the $H_{(j)}$'s and additionally are bounded by n. Hence $H1_{[0,T_n]}$ is an elementary process, and by the very definition of the stochastic integral with respect to a martingale of $\mathbb{H}^2$, we have

$$(H \cdot M)_{t \wedge T_n} = (H1_{[0,T_n]} \cdot M)_t = \sum_{j=0}^{p-1} H^n_{(j)} (M_{t_{j+1} \wedge t} - M_{t_j \wedge t})$$

The desired result now follows by letting n tend to infinity.