Math 562 Fall 2020

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Outline

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I posted HW4 in my homepage. HW4 is due 10/16 at noon. I also set up HW4 in the course Moodle page.

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Recall that $\mathcal{P}$ is the progressive σ-field on $\Omega \times \mathbb{R}_+$. If $M \in \mathbb{H}^2$, we use $L^2(M)$ to denote the collection of all progressive processes H such that

$$\mathbb{E}\left[\int_0^\infty H_s^2 d\langle M, M\rangle_s\right] < \infty$$

with the convention that two progressive processes H and H' satisfying this integrability condition are identified if $H_s = H'_s$, $d\langle M, M\rangle_s$ a.e. a.s.

We can view $L^2(M)$ as an ordinary L^2 space, namely

$$L^2(M) = L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, d\mathbb{P}d\langle M, M\rangle_s),$$

where $d\mathbb{P}d\langle M, M\rangle_s$ refers to the finite measure on $(\Omega \times \mathbb{R}_+, \mathcal{P})$ that assigns the mass

$$\mathbb{E}\left[\int_0^\infty 1_A(\omega, s) d\langle M, M\rangle_s\right]$$

to a set $A \in \mathcal{P}$.

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Just like any L^2 space, the space $L^2(M)$ is a Hilbert space for the scalar product

$$(H, K)_{L^2(M)} = \mathbb{E}\left[\int_0^\infty H_s K_s d\langle M, M\rangle_s\right]$$

and the associated norm is

$$\|H\|_{L^2(M)} = \left(\mathbb{E}\left[\int_0^\infty H_s^2 d\langle M, M\rangle_s\right]\right).$$

Definition 5.2

An elementary process is a progressive process of the form

$$H_s(\omega) = \sum_{j=0}^{p-1} H_{(j)}(\omega) 1_{(t_j, t_{j+1}]}(s)$$

where $0 = t_0 < t_1 < \cdots < t_p = t$ and for every $j = 0, \ldots, p-1$, $H_{(j)}$ is a bounded $\mathcal{F}_{t_j}$-measurable random variable.

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The family $\mathcal{E}$ of all elementary processes forms a linear subspace of $L^2(M)$.

Proposition 5.3

For every $M \in \mathbb{H}^2$, $\mathcal{E}$ is dense in $L^2(M)$.

Proof of Proposition 5.3

It suffices to show that, if $K \in L^2(M)$ is orthogonal to $\mathcal{E}$, then $K = 0$. Assume that $K \in L^2(M)$ is orthogonal to $\mathcal{E}$, and define, for every $t \geq 0$,

$$X_t = \int_0^t K_u d\langle M, M\rangle_u.$$

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Proof of Proposition 5.3 (cont)

To see that the integral in the right-hand side makes sense, and defines a finite variation process $(X_t)_{t\geq 0}$, we use the Cauchy-Schwarz inequality to get that

$$\mathbb{E}\left[\int_0^t |K_u| d\langle M, M\rangle_u\right] \leq \left(\mathbb{E}\left[\int_0^t K_u^2 d\langle M, M\rangle_u\right]\right)^{1/2} \cdot (\mathbb{E}[\langle M, M\rangle_\infty])^{1/2}.$$

The right-hand side is finite since $M \in \mathbb{H}^2$ and $K \in L^2(M)$, and thus we have in particular

$$a.s. \quad \forall t \geq 0, \quad \int_0^t |K_u| d\langle M, M\rangle_u < \infty.$$

Thus $(X_t)_{t\geq 0}$ is well defined as a finite variation process. The preceding bound also shows that $X_t \in L^1$ for every $t \geq 0$.

Proof of Proposition 5.3 (cont)

Let $0 \leq s < t$, let F be a bounded $\mathcal{F}_s$-measurable random-variable, and let $H \in \mathcal{E}$ be the elementary process defined by $H_r(\omega) = F(\omega)1_{(s,t]}(r)$. Since $(H, K)_{L^2(M)} = 0$, we have

$$\mathbb{E}\left[F \int_s^t K_u d\langle M, M\rangle_u\right] = 0.$$

It follows that $\mathbb{E}[F(X_t - X_s)] = 0$ for every $s < t$ and every bounded $\mathcal{F}_s$-measurable random-variable F. Since the process X is adapted and we know that $X_t \in L^1$ for every $t \geq 0$, this implies that X is a (continuous) martingale. On the other hand, X is also a finite variation process and, by Theorem 4.8, this is only possible if $X = 0$.

Proof of Proposition 5.3 (cont)

We have thus proved that

$$\int_0^t K_u d\langle M, M\rangle_u = 0, \quad \forall t \geq 0, a.s.$$

which implies that, a.s., the signed measure having density K_u with respect to $d\langle M, M\rangle_u$ is the zero measure, which is only possible if

$$K_u = 0, \quad d\langle M, M\rangle_u - a.e. \quad a.s.$$

or, equivalently $K = 0$ in $L^2(M)$.

Recall that, for stopping time T, X^T stands for the stopped process $X_t^T = X_{t\wedge T}$. If $M \in \mathbb{H}^2$, the fact that $\langle M^T, M^T\rangle_\infty = \langle M, M\rangle_T$ immediately implies that $M^T \in \mathbb{H}^2$. Furthermore, if $H \in L^2(M)$, the process $1_{[0,T]}H$ defined by $(1_{[0,T]}H)_s(\omega) = 1_{\{0\leq s\leq T(\omega)\}}H_s(\omega)$ also belongs to $L^2(M)$.

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Theorem 5.4

Let $M \in \mathbb{H}^2$. For every $H \in \mathcal{E}$ of the form

$$H_s(\omega) = \sum_{j=0}^{p-1} H_{(j)}(\omega) 1_{(t_j, t_{j+1}]}(s).$$

the formula

$$(H \cdot M)_t = \sum_{j=0}^{p-1} H_{(j)}(M_{t_{j+1} \wedge t} - M_{t_j \wedge t})$$

defines a process $H \cdot M \in \mathbb{H}^2$. The mapping $H \mapsto H \cdot M$ extends to an isometry from $L^2(M)$ to $\mathbb{H}^2$. Furthermore, $H \cdot M$ is the unique martingale of $\mathbb{H}^2$ with the property

$$\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle, \quad \forall N \in \mathbb{H}^2. \tag{1}$$

If T is a stopping time, we have

$$(1_{[0,T]} H) \cdot M = (H \cdot M)^T = H \cdot M^T. \tag{2}$$

We often use the notation

$$(H \cdot M)_t = \int_0^t H_s dM_s$$

and call $H \cdot M$ the stochastic integral of H with respect to M.

The quantity $H \cdot \langle M, N \rangle$ in the right-hand side of (1) is an integral with respect to a finite variation process. The fact that we use a similar notation $H \cdot A$ and $H \cdot M$ for the integrals with respect to a finite variation process A and with respect to a martingale M creates no ambiguity since these two classes of processes are essentially disjoint.

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Proof of Theorem 5.4

First note that the definition of $H \cdot M$ does not depend on the decomposition chosen for H in the first display of the theorem. Using this remark, one then checks that the mapping $H \mapsto H \cdot M$ is linear.

Now we verify that this mapping is an isometry from $\mathcal{E}$ (viewed as a subspace of $L^2(M)$) into $\mathbb{H}^2$.

Fix $H \in \mathcal{E}$ of the form given in the theorem, and for every $j \in \{0, 1, \ldots, p-1\}$, define

$$M_t^j = H_{(j)}(M_{t_{j+1} \wedge t} - M_{t_j \wedge t}),$$

for every $t \geq 0$. It is easy to check that M^j is a martingale and this martingale belongs to $\mathbb{H}^2$. It follows that $H \cdot M = \sum_{j=0}^{p-1} M^j$ is also a martingale in $\mathbb{H}^2$. Note that the continuous martingales M^j are orthogonal, and their respective quadratic variations are given by

$$\langle M^j, M^j \rangle_t = H_{(j)}^2(\langle M, M \rangle_{t_{j+1} \wedge t} - \langle M, M \rangle_{t_j \wedge t}).$$

Proof of Theorem 5.4 (cont)

Thus

$$\langle H\cdot M, H\cdot M\rangle_t=\sum_{j=0}^{p-1}H_{(j)}^2(\langle M,M\rangle_{t_{j+1}\wedge t}-\langle M,M\rangle_{t_j\wedge t})=\int_0^t H_s^2 d\langle M,M\rangle_s.$$

Consequently,

$$\|H\cdot M\|_{\mathbb{H}^2}^2=\mathbb{E}[\langle H\cdot M,H\cdot M\rangle_\infty]=E[\int_0^\infty H_s^2 d\langle M,M\rangle_s]=\|H\|_{L^2(M)}^2.$$

By linearity, this implies that $H\cdot M=H'\cdot M$ if H' is another elementary process that is identified with H in $L^2(M)$. Therefore the mapping $H\mapsto H\cdot M$ makes sense from $\mathcal{E}$ (viewed as a subspace of $L^2(M)$) into $\mathbb{H}^2$. This latter mapping is linear, and, since it preserves the norm, it is an isometry from $\mathcal{E}$ (viewed as a subspace of $L^2(M)$) into $\mathbb{H}^2$. Since $\mathcal{E}$ is dense in $L^2(M)$ and $\mathbb{H}^2$ is a Hilbert space, this mapping can be extended in a unique way to an isometry from $L^2(M)$ into $\mathbb{H}^2$.

Proof of Theorem 5.4 (cont)

Let us now prove (1). Fix $N \in \mathbb{H}^2$. We first note that, if $H \in L^2(M)$, the Kunita-Watanabe inequality shows that

$$\mathbb{E}\left[\int_0^\infty |H_s||d\langle M, N\rangle_s|\right] \leq \|H\|_{L^2(M)}\|N\|_{\mathbb{H}^2} < \infty$$

and thus the variable $\int_0^\infty H_s d\langle M, N\rangle_s = (H \cdot \langle M, N\rangle)_\infty$ is well defined and in L^1. Consider first the case where H is an elementary process of the form given in the theorem, and define the continuous martingales M^j, $j = 0, \ldots p-1$, as before. Then,

$$\langle H \cdot M, N\rangle = \sum_{j=0}^{p-1} \langle M^j, N\rangle$$

and for every $j = 0, \ldots p-1$,

$$\langle M^j, N\rangle_t = H_{(j)}(\langle M, N\rangle_{t_{j+1}\wedge t} - \langle M, N\rangle_{t_j \wedge t}).$$

Proof of Theorem 5.4 (cont)

It follows that

$$\langle H \cdot M, N \rangle_t = \sum_{j=0}^{p-1} H_{(j)}(\langle M, N \rangle_{t_{j+1} \wedge t} - \langle M, N \rangle_{t_j \wedge t}) = \int_0^t H_s d\langle M, N \rangle_s.$$

which gives (1) when $H \in \mathcal{E}$. We then observe that the linear mapping $X \mapsto \langle X, N \rangle_\infty$ is continuous from $\mathbb{H}^2$ into L^1. Indeed, by the Kunita-Watanabe inequality,

$$\mathbb{E}[\langle X, N \rangle_\infty] \leq \mathbb{E}[\langle X, X \rangle_\infty]^{1/2} \mathbb{E}[\langle N, N \rangle_\infty]^{1/2} = \|X\|_{\mathbb{H}^2} \|N\|_{\mathbb{H}^2}.$$

If $(H^n)_{n \geq 1}$ is a sequence in $\mathcal{E}$ such that $H^n \to H$ in $L^2(M)$, we have therefore

$$\langle H \cdot M, N \rangle_\infty = \lim_{n \to \infty} \langle H^n \cdot M, N \rangle_\infty = \lim_{n \to \infty} (H^n \cdot \langle M, N \rangle)_\infty = (H \cdot \langle M, N \rangle)_\infty.$$

where the convergences hold in L^1, and the last equality again follows from the Kunita-Watanabe inequality by writing

Proof of Theorem 5.4 (cont)

$$\mathbb{E}\left[\int_0^\infty (H_s^n - H_s)d\langle M, N\rangle_s\right] \leq \mathbb{E}[\langle N, N\rangle_\infty]^{1/2}\|H^n - H\|_{L^2(M)}.$$

We have thus obtained the identity $\langle H \cdot M, N\rangle_\infty = \langle H \cdot M, N\rangle_\infty$, but replacing N by the stopped martingale N^t in this identity also gives $\langle H \cdot M, N\rangle_t = \langle H \cdot M, N\rangle_t$, which completes the proof of (1).

It is easy to see that (1) characterizes $H \cdot M$ among the martingales of $\mathbb{H}^2$. Indeed, if X is another martingale of $\mathbb{H}^2$ hat satisfies the same identity, we get, for every $N \in \mathbb{H}^2$

$$\langle H \cdot M - X, N\rangle = 0.$$

Taking $N = H \cdot M - X$ and using Proposition 4.12 we obtain that $X = H \cdot M$.

Proof of Theorem 5.4 (cont)

It remains to verify (2). Using the properties of the bracket of two continuous local martingales, we observe that, if $N \in \mathbb{H}^2$,

$$\langle H \cdot M^T, N\rangle_t = \langle H \cdot M, N\rangle_{t\wedge T} = (H \cdot \langle M, N\rangle)_{t\wedge T} = (1_{[0,T]}H \cdot \langle M, N\rangle)_t$$

which shows that the stopped martingale $(H \cdot M)^T$ satisfies the characteristic property of the stochastic integral $(1_{[0,T]}H) \cdot M$. The first equality in (2) follows. The second one is proved analogously, writing

$$\langle H \cdot M^T, N\rangle = H \cdot \langle M^T, N\rangle = H \cdot \langle M, N\rangle^T = 1_{[0,T]}H \cdot \langle M, N\rangle.$$

This completes the proof of the theorem.

Remark

We could have used (1) to define the stochastic integral $H \cdot M$, observing that the mapping $N \mapsto \mathbb{E}[(H \cdot \langle M, N \rangle)_\infty]$ yields a continuous linear map on $\mathbb{H}^2$, and thus there exists a unique martingale $H \cdot M$ in $\mathbb{H}^2$ such that

$$\mathbb{E}[(H \cdot \langle M, N \rangle)_\infty] = (H \cdot M, N)_{\mathbb{H}^2} = \mathbb{E}[(\langle H \cdot M, N \rangle)_\infty].$$

Using the notation introduced after Theorem 5.4, we can rewrite (1) in the form

$$\langle \int_0^\cdot H_s dM_s, N \rangle_t = \int_0^t H_s d\langle M, N \rangle_s$$

We interpret this by saying that the stochastic integral "commutes" with the bracket. Let us immediately mention a very important consequence.

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We interpret this by saying that the stochastic integral “commutes” with the bracket. Let us immediately mention a very important consequence.

If $M \in \mathbb{H}^2$ and $H \in L^2(M)$, two successive applications of (1) give

$$\langle H \cdot M, H \cdot M \rangle = H \cdot (H \cdot \langle M, M \rangle) = H^2 \cdot \langle M, M \rangle$$

using the "associativity property" integrals with respect to finite variation processes. Put differently, the quadratic variation of the continuous martingale $H \cdot M$ is

$$\langle \int_0^{\cdot} H_s dM_s, \int_0^{\cdot} H_s dM_s \rangle_t = \int_0^t H_s^2 d\langle M, M \rangle_s.$$

More generally, if N is another martingale of $\mathbb{H}^2$ and $K \in L^2(N)$, the same argument gives

$$\langle \int_0^{\cdot} H_s dM_s, \int_0^{\cdot} K_s dN_s \rangle_t = \int_0^t H_s K_s d\langle M, N \rangle_s.$$