Math 562 Fall 2020

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Outline

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I will post HW4 later today. HW4 is due 10/16.

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Proposition 4.18 (Kunita-Watanabe)

Let M and N be two continuous local martingales and let H and K be two measurable processes. Then, a.s.,

$$\int_0^\infty |H_s||K_s||d\langle M,N\rangle_s| \le \left(\int_0^\infty H_s^2 d\langle M,M\rangle_s\right)^{1/2}\left(\int_0^\infty K_s^2 d\langle N,N\rangle_s\right)^{1/2}$$

Proof of Proposition 4.18

In this proof, we use the special notation $\langle M,N\rangle_s^t = \langle M,N\rangle_t - \langle M,N\rangle_s$ for $0 \le s \le t$. The first step of the proof is to observe that we have a.s. for every choice of the rationals $s < t$ (and also by continuity for every reals $s < t$)

$$|\langle M,N\rangle_s^t| \le \sqrt{\langle M,M\rangle_s^t}\sqrt{\langle N,N\rangle_s^t}.$$

Indeed, this follows from the approximations of $\langle M,M\rangle$ and $\langle M,N\rangle$ in Theorem 4.9 and in Proposition 4.15 respectively, together with the Cauchy-Schwarz inequality.

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Indeed, this follows from the approximations of $\langle M, M\rangle$ and $\langle M, N\rangle$ in Theorem 4.9 and in Proposition 4.15 respectively, together with the Cauchy-Schwarz inequality.

Proof of Proposition 4.18 (cont)

From now on, we fix ω such that the inequality of the last display holds for every $s < t$, and we argue with this value ω.

Note also that, for every $s < t$,

$$\int_s^t |d\langle M, N\rangle_u| \leq \sqrt{\langle M, M\rangle_s^t}\sqrt{\langle N, N\rangle_s^t}. \tag{1}$$

Indeed, we use Proposition 4.2, noting that, for any partition $s = t_0 < t_1 < \cdots < t_p = t$, we have

$$\begin{aligned}\sum_{j=1}^{p} |\langle M, N\rangle_{t_{j-1}}^{t_j}| &\leq \sum_{j=1}^{p} \sqrt{\langle M, M\rangle_{t_{j-1}}^{t_j}}\sqrt{\langle N, N\rangle_{t_{j-1}}^{t_j}} \\ &\leq \left(\sum_{j=1}^{p} \langle M, M\rangle_{t_{j-1}}^{t_j}\right)^{1/2} \left(\sum_{j=1}^{p} \langle N, N\rangle_{t_{j-1}}^{t_j}\right)^{1/2} \\ &= \sqrt{\langle M, M\rangle_s^t}\sqrt{\langle N, N\rangle_s^t}.\end{aligned}$$

Proof of Proposition 4.18 (cont)

We then get that, for every bounded Borel subset A of $\mathbb{R}_+$,

$$\int_A |d\langle M,N\rangle_u| \leq \sqrt{\int_A d\langle M,M\rangle_u}\sqrt{\int_A d\langle N,N\rangle_u}.$$

When $A = [s,t]$, this is (1). If A is a finite union of intervals, this follows from (1) and another application of the Cauchy-Schwarz inequality. A monotone class argument shows that the inequality of the last display remains valid for any bounded Borel set A.

Next let $h = \sum_{j=1}^{p} \lambda_j 1_{A_j}$ and $k = \sum_{j=1}^{p} \mu_j 1_{A_j}$ be two non-negative simple functions on $\mathbb{R}_+$ with bounded support contained in $[0,K]$, for some $K > 0$. Here $A_1,\ldots,A_p$ is a measurable partition of $[0,K]$, and $\lambda_1,\ldots,\lambda_p,\mu_1,\ldots\mu_p$ are non-negative reals. Then

Proof of Proposition 4.18 (cont)

$$\begin{aligned}\int h(s)k(s)|d\langle M,N\rangle_s| &= \sum_{j=1}^{p} \lambda_j \mu_j \int_{A_j} |d\langle M,N\rangle_s| \\ &\le \left(\sum_{j=1}^{p} \lambda_j^2 d\langle M,M\rangle_s\right)^{1/2} \left(\sum_{j=1}^{p} \mu_j^2 d\langle N,N\rangle_s\right)^{1/2} \\ &= \left(\int h^2(s) d\langle M,M\rangle_s\right)^{1/2} \left(\int k^2(s) d\langle N,N\rangle_s\right)^{1/2},\end{aligned}$$

which gives the desired inequality for simple functions. Since every non-negative Borel function is a monotone increasing limit of simple functions with bounded support, an application of the monotone convergence theorem completes the proof.

Outline

Definition 4.19

A process $X = (X_t)_{t \geq 0}$ is a continuous semimartingale if it can be written in the form

$$X_t = M_t + A_t$$

where M is a continuous local martingale and A is a finite variation process.

The decomposition $X = M + A$ is then unique up to indistinguishability thanks to Theorem 4.8. We say that this is the canonical decomposition of X.

By construction, continuous semimartingales have continuous sample paths. It is possible to define a notion of semimartingale with cadla sample paths, but we will only deal with continuous semimartingales, and for this reason we sometimes omit the word continuous.

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Definition 4.20

Let $X = M + A$ and $Y = M' + A'$ be the canonical decompositions of two continuous semimartingales X and Y. The bracket $\langle X, Y\rangle$ is the finite variation process defined by

$$\langle X, Y\rangle_t = \langle M, M'\rangle_t.$$

In particular, we have $\langle X, X\rangle_t = \langle M, M\rangle_t$.

Proposition 4.21

Let $0 = t_0^n < t_1^n < \cdots < t_p^n = t$ be an increasing sequence of partitions of $[0, t]$ whose mesh tends to 0. Then,

$$\lim_{n\to\infty} \sum_{j=1}^{p_n} (X_{t_j^n} - X_{t_{j-1}^n})(Y_{t_j^n} - Y_{t_{j-1}^n}) = \langle X, Y\rangle_t$$

in probability.

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in probability.

Proof Proposition 4.21

We treat the case $X = Y$ only. The general case is similar. We have

$$\sum_{j=1}^{p_n}(X_{t_j^n} - X_{t_{j-1}^n})^2 = \sum_{j=1}^{p_n}(M_{t_j^n} - M_{t_{j-1}^n})^2 + \sum_{j=1}^{p_n}(A_{t_j^n} - A_{t_{j-1}^n})^2 + 2\sum_{j=1}^{p_n}(M_{t_j^n} - M_{t_{j-1}^n})(A_{t_j^n} - A_{t_{j-1}^n}).$$

By Theorem 4.9,

$$\lim_{n\to\infty}\sum_{j=1}^{p_n}(M_{t_j^n} - M_{t_{j-1}^n}) = \langle M, M\rangle_t = \langle X, X\rangle_t$$

in probability. On the other hand,

Proof Proposition 4.21 (cont)

$$\sum_{j=1}^{p_n}(A_{t_j^n} - A_{t_{j-1}^n})^2 \leq \left(\sup_{1\leq j\leq p_n} |A_{t_j^n} - A_{t_{j-1}^n}|\right) \sum_{j=1}^{p_n} |A_{t_j^n} - A_{t_{j-1}^n}|$$
$$\leq \left(\int_0^t |dA_s|\right) \sup_{1\leq j\leq p_n} |A_{t_j^n} - A_{t_{j-1}^n}|$$

which tends to 0 a.s. when $n \to \infty$ by the continuity of sample paths of A. The same argument shows that

$$|\sum_{j=1}^{p_n}(M_{t_j^n} - M_{t_{j-1}^n})(A_{t_j^n} - A_{t_{j-1}^n})| \leq \left(\int_0^t |dA_s|\right) \sup_{1\leq j\leq p_n} |M_{t_j^n} - M_{t_{j-1}^n}|$$

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Outline

Throughout this chapter, we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and assume that $(\mathcal{F}_t)_{t\geq 0}$ is complete. We write $\mathbb{H}^2$ for the space of all continuous martingales M which are bounded in L^2 such that $M_0 = 0$, with the usual convention that two indistinguishable processes are identified. Equivalently, $M \in \mathbb{H}^2$ if and only M is a continuous local martingale such that $M_0 = 0$ and $\mathbb{E}[\langle M, M\rangle_\infty] < \infty$.

If $M \in \mathbb{H}^2$, then $M_t = \mathbb{E}[M_\infty|\mathcal{F}_t]$ where $M_\infty \in L^2$ is the almost sure limit of M_t as $t \to \infty$. If $M, N \in \mathbb{H}^2$, then $\langle M, N\rangle_\infty$ is well defined and $\mathbb{E}[|\langle M, N\rangle_\infty|] < \infty$. This allows us to define a symmetric bilinear form on $\mathbb{H}^2$ via the formula

$$(M, N)_{\mathbb{H}^2} = \mathbb{E}[\langle M, N\rangle_\infty] = \mathbb{E}[M_\infty N_\infty],$$

where the second equality comes from Proposition 4.15 (v).

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where the second equality comes from Proposition 4.15 (v).

Clearly $(M, M)_{\mathbb{H}^2} = 0$ if and only $M = 0$. The scalar product $(M, N)_{\mathbb{H}^2}$ thus yields a norm on $\mathbb{H}^2$ given by

$$\|M\|_{\mathbb{H}^2} = (M, M)_{\mathbb{H}^2}^{1/2} = \mathbb{E}[\langle M, M\rangle_\infty]^{1/2} = \mathbb{E}[M_\infty M_\infty]^{1/2}.$$

Proposition 5.1

The space $\mathbb{H}^2$ equipped with the scalar product $(M, N)_{\mathbb{H}^2}$ is a Hilbert space.

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Proposition 5.1

The space $\mathbb{H}^2$ equipped with the scalar product $(M, N)_{\mathbb{H}^2}$ is a Hilbert space.

Proof of Proposition 5.1

We need to verify that the vector space $\mathbb{H}^2$ is complete for the norm $||M||_{\mathbb{H}^2}$. Let $(M^n)_{n\geq 1}$ be a Cauchy sequence in $\mathbb{H}^2$. We have then

$$\lim_{m,n\to\infty} \mathbb{E}[(M^n_\infty - M^m_\infty)^2] = \lim_{m,n\to\infty} (M^n - M^m, M^n - M^m)_{\mathbb{H}^2} = 0.$$

Consequently, the sequence $(M^n_\infty)_{n\geq 1}$ converges in L^2 to a limit, which we denote by Z. On the other hand, Doob's L^2-inequality and a straightforward passage to the limit show that, for every m, n,

$$\mathbb{E}[\sup_{t\geq 0}(M^n_t - M^m_t)^2] \leq 4\mathbb{E}[(M^n_\infty - M^m_\infty)^2].$$

Thus

$$\lim_{m,n\to\infty} \mathbb{E}[\sup_{t\geq 0}(M^n_t - M^m_t)^2] = 0. \qquad (2)$$

Hence, for every $t > 0$, M^n_t converges in L^2, and we want to argue that the limit yields a process with continuous sample paths.

Proof of Proposition 5.1 (cont)

To this end, we use (2) to find an increasing sequence $n_k \uparrow \infty$ such that

$$\mathbb{E}\left[\sum_{k=1}^{\infty} \sup_{t\geq 0} |M_t^{n_k} - M_t^{n_{k+1}}|\right] \leq \sum_{k=1}^{\infty} \mathbb{E}\left[\sup_{t\geq 0} (M_t^{n_k} - M_t^{n_{k+1}})^2\right]^{1/2} < \infty.$$

Thus we have a.s.

$$\sum_{k=1}^{\infty} \sup_{t\geq 0} |M_t^{n_k} - M_t^{n_{k+1}}| < \infty.$$

and thus the sequence $(M_t^{n_k})_{t\geq 0}$ converges uniformly on $\mathbb{R}_+$ a.s., to a limit denoted by $(M_t)_{t\geq 0}$. On the zero probability set where the uniform convergence does not hold, we take $M_t = 0$ for every $t > 0$.

Proof of Proposition 5.1 (cont)

Clearly the limiting process M has continuous sample paths and is adapted. Furthermore, from the L^2-convergence of (M_∞^n) to Z we immediately get by passing to the limit in the identity $M_t^{n_k} = \mathbb{E}[M_\infty^{n_k} | \mathcal{F}_t]$ that $M_t = \mathbb{E}[Z | \mathcal{F}_t]$. Hence $(M_t)_{t \geq 0}$ is a continuous martingale and is bounded in L^2, so that $M \in \mathbb{H}^2$. The a.s. uniform convergence of $(M_t^{n_k})_{t \geq 0}$ to $(M_t)_{t \geq 0}$ then ensures that $M_\infty = \lim_{k \to \infty} M_\infty^{n_k} = Z$ a.s. Finally, the L^2-convergence of (M_∞^n) to $Z = M_\infty$ shows that the sequence (M^n) converges to M in $\mathbb{H}^2$.