I will post HW4 later today. HW4 is due 10/16.
Outline

1. General Info
2. 4.4 The Bracket of Two Continuous Local Martingales
3. 4.5 Continuous Semimartingales
4. 5.1 The Construction of Stochastic Integrals
Proposition 4.18 (Kunita-Watanabe)

Let $M$ and $N$ be two continuous local martingales and let $H$ and $K$ be two measurable processes. Then, a.s.,

$$
\int_0^\infty |H_s||K_s||d\langle M, N\rangle_s| \leq \left(\int_0^\infty H_s^2 d\langle M, M\rangle_s\right)^{1/2} \left(\int_0^\infty K_s^2 d\langle N, N\rangle_s\right)^{1/2}
$$

Proof of Proposition 4.18

In this proof, we use the special notation $\langle M, N\rangle_t^s = \langle M, N\rangle_t - \langle M, N\rangle_s$ for $0 \leq s \leq t$. The first step of the proof is to observe that we have a.s. for every choice of the rationals $s < t$ (and also by continuity for every reals $s < t$)

$$
|\langle M, N\rangle_t^s| \leq \sqrt{\langle M, M\rangle_t^s} \sqrt{\langle N, N\rangle_t^s}.
$$

Indeed, this follows from the approximations of $\langle M, M\rangle$ and $\langle M, N\rangle$ in Theorem 4.9 and in Proposition 4.15 respectively, together with the Cauchy-Schwarz inequality.
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Indeed, this follows from the approximations of $\langle M, M \rangle$ and $\langle M, N \rangle$ in Theorem 4.9 and in Proposition 4.15 respectively, together with the Cauchy-Schwarz inequality.
Proof of Proposition 4.18 (cont)

From now on, we fix \( \omega \) such that the inequality of the last display holds for every \( s < t \), and we argue with this value \( \omega \).

Note also that, for every \( s < t \),

\[
\int_s^t |d\langle M, N \rangle_u| \leq \sqrt{\langle M, M \rangle_s^t} \sqrt{\langle N, N \rangle_s^t}.
\]  

(1)

Indeed, we use Proposition 4.2, noting that, for any partition \( s = t_0 < t_1 < \cdots < t_p = t \), we have

\[
\sum_{j=1}^p \left| \langle M, N \rangle_{t_j}^{t_{j-1}} \right| \leq \sum_{j=1}^p \sqrt{\langle M, M \rangle_{t_j}^{t_{j-1}}} \sqrt{\langle N, N \rangle_{t_j}^{t_{j-1}}}
\]

\[
\leq \left( \sum_{j=1}^p \langle M, M \rangle_{t_j}^{t_{j-1}} \right)^{1/2} \left( \sum_{j=1}^p \langle N, N \rangle_{t_j}^{t_{j-1}} \right)^{1/2}
\]

\[
= \sqrt{\langle M, M \rangle_s^t} \sqrt{\langle N, N \rangle_s^t}.
\]
Proof of Proposition 4.18 (cont)

We then get that, for every bounded Borel subset $A$ of $\mathbb{R}_+$,

$$\int_A |d\langle M, N \rangle_u| \leq \sqrt{\int_A d\langle M, M \rangle_u} \sqrt{\int_A d\langle N, N \rangle_u}.$$ 

When $A = [s, t]$, this is (1). If $A$ is a finite union of intervals, this follows from (1) and another application of the Cauchy-Schwarz inequality. A monotone class argument shows that the inequality of the last display remains valid for any bounded Borel set $A$.

Next let $h = \sum_{j=1}^p \lambda_j 1_{A_j}$ and $k = \sum_{j=1}^p \mu_j 1_{A_j}$ be two non-negative simple functions on $\mathbb{R}_+$ with bounded support contained in $[0, K]$, for some $K > 0$. Here $A_1, \ldots, A_p$ is a measurable partition of $[0, K]$, and $\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_p$ are non-negative reals. Then
Proof of Proposition 4.18 (cont)

\[ \int h(s)k(s)\left|d\langle M, N\rangle_s\right| = \sum_{j=1}^{p} \lambda_j \mu_j \int_{A_j} \left|d\langle M, N\rangle_s\right| \]

\[ \leq \left( \sum_{j=1}^{p} \lambda_j^2 d\langle M, M\rangle_s \right)^{1/2} \left( \sum_{j=1}^{p} \mu_j^2 d\langle N, N\rangle_s \right)^{1/2} \]

\[ = \left( \int h^2(s)d\langle M, M\rangle_s \right)^{1/2} \left( \int k^2(s)d\langle N, N\rangle_s \right)^{1/2}, \]

which gives the desired inequality for simple functions. Since every non-negative Borel function is a monotone increasing limit of simple functions with bounded support, an application of the monotone convergence theorem completes the proof.
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Definition 4.19

A process $X = (X_t)_{t \geq 0}$ is a continuous semimartingale if it can be written in the form

$$X_t = M_t + A_t$$

where $M$ is a continuous local martingale and $A$ is a finite variation process.

The decomposition $X = M + A$ is then unique up to indistinguishability thanks to Theorem 4.8. We say that this is the canonical decomposition of $X$.

By construction, continuous semimartingales have continuous sample paths. It is possible to define a notion of semimartingale with cadlag sample paths, but we will only deal with continuous semimartingales, and for this reason we sometimes omit the word continuous.
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Definition 4.20

Let $X = M + A$ and $Y = M' + A'$ be the canonical decompositions of two continuous semimartingales $X$ and $Y$. The bracket $\langle X, Y \rangle$ is the finite variation process defined by

$$\langle X, Y \rangle_t = \langle M, M' \rangle_t.$$ 

In particular, we have $\langle X, X \rangle_t = \langle M, M \rangle_t$.

Proposition 4.21

Let $0 = t^n_0 < t^n_1 < \cdots < t^n_p = t$ be an increasing sequence of partitions of $[0, t]$ whose mesh tends to 0. Then,

$$\lim_{n \to \infty} \sum_{j=1}^{p_n} (X^n_{t^n_j} - X^n_{t^n_{j-1}})(Y^n_{t^n_j} - Y^n_{t^n_{j-1}}) = \langle X, Y \rangle_t$$

in probability.
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In particular, we have $\langle X, X \rangle_t = \langle M, M \rangle_t$.

**Proposition 4.21**

Let $0 = t_0^n < t_1^n < \cdots < t_p^n = t$ be an increasing sequence of partitions of $[0, t]$ whose mesh tends to $0$. Then,

$$\lim_{n \to \infty} \sum_{j=1}^{p_n} (X_{t_j^n} - X_{t_{j-1}^n})(Y_{t_j^n} - Y_{t_{j-1}^n}) = \langle X, Y \rangle_t$$

in probability.
**Definition 4.20**

Let $X = M + A$ and $Y = M' + A'$ be the canonical decompositions of two continuous semimartingales $X$ and $Y$. The bracket $\langle X, Y \rangle$ is the finite variation process defined by

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In particular, we have $\langle X, X \rangle_t = \langle M, M \rangle_t$.

**Proposition 4.21**

Let $0 = t_0^n < t_1^n < \cdots < t_{\rho_n}^n = t$ be an increasing sequence of partitions of $[0, t]$ whose mesh tends to 0. Then,

$$\lim_{n \to \infty} \sum_{j=1}^{\rho_n} (X_{t_j^n} - X_{t_{j-1}^n})(Y_{t_j^n} - Y_{t_{j-1}^n}) = \langle X, Y \rangle_t$$

in probability.
Proof Proposition 4.21

We treat the case $X = Y$ only. The general case is similar. We have

$$
\sum_{j=1}^{p_n} (X^n_{t_j} - X^n_{t_{j-1}})^2 = \sum_{j=1}^{p_n} (M^n_{t_j} - M^n_{t_{j-1}})^2 + \sum_{j=1}^{p_n} (A^n_{t_j} - A^n_{t_{j-1}})^2
$$

$$+ 2 \sum_{j=1}^{p_n} (M^n_{t_j} - M^n_{t_{j-1}})(A^n_{t_j} - A^n_{t_{j-1}}).$$

By Theorem 4.9,

$$\lim_{n \to \infty} \sum_{j=1}^{p_n} (M^n_{t_j} - M^n_{t_{j-1}}) = \langle M, M \rangle_t = \langle X, X \rangle_t$$

in probability. On the other hand,
Proof Proposition 4.21 (cont)

\[
\frac{p_n}{\sum_{j=1}^{p_n} (A_{t_j}^n - A_{t_{j-1}}^n)^2} \leq \left( \sup_{1 \leq j \leq p_n} |A_{t_j}^n - A_{t_{j-1}}^n| \right) \sum_{j=1}^{p_n} |A_{t_j}^n - A_{t_{j-1}}^n| \leq \left( \int_0^t |dA_s| \right) \sup_{1 \leq j \leq p_n} |A_{t_j}^n - A_{t_{j-1}}^n| \]

which tends to 0 a.s. when \( n \to \infty \) by the continuity of sample paths of \( A \). The same argument shows that

\[
\left| \sum_{j=1}^{p_n} (M_{t_j}^n - M_{t_{j-1}}^n)(A_{t_j}^n - A_{t_{j-1}}^n) \right| \leq \left( \int_0^t |dA_s| \right) \sup_{1 \leq j \leq p_n} |M_{t_j}^n - M_{t_{j-1}}^n| \]

tends to 0 a.s.
Outline

1. General Info
2. 4.4 The Bracket of Two Continuous Local Martingales
3. 4.5 Continuous Semimartingales
4. 5.1 The Construction of Stochastic Integrals
Throughout this chapter, we fix a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) and assume that \((\mathcal{F}_t)_{t \geq 0}\) is complete. We write \(H^2\) for the space of all continuous martingales \(M\) which are bounded in \(L^2\) such that \(M_0 = 0\), with the usual convention that two indistinguishable processes are identified. Equivalently, \(M \in H^2\) if and only if \(M\) is a continuous local martingale such that \(M_0 = 0\) and \(\mathbb{E}[\langle M, M \rangle_\infty] < \infty\).

If \(M \in H^2\), then \(M_t = \mathbb{E}[M_\infty | \mathcal{F}_t]\) where \(M_\infty \in L^2\) is the almost sure limit of \(M_t\) as \(t \to \infty\). If \(M, N \in H^2\), then \(\langle M, N \rangle_\infty\) is well defined and \(\mathbb{E}[|\langle M, N \rangle_\infty|] < \infty\). This allows us to define a symmetric bilinear form on \(H^2\) via the formula

\[(M, N)_{H^2} = \mathbb{E}[\langle M, N \rangle_\infty] = \mathbb{E}[M_\infty N_\infty],\]

where the second equality comes from Proposition 4.15 (v).
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$$(M, N)_{\mathbb{H}^2} = \mathbb{E}[\langle M, N \rangle_\infty] = \mathbb{E}[M_\infty N_\infty],$$

where the second equality comes from Proposition 4.15 (v).
Clearly \((M, M)_{\mathcal{H}^2} = 0\) if and only \(M = 0\). The scalar product \((M, N)_{\mathcal{H}^2}\) thus yields a norm on \(\mathcal{H}^2\) given by

\[
\|M\|_{\mathcal{H}^2} = (M, M)^{1/2} = \mathbb{E}[(M, M)_{\infty}]^{1/2} = \mathbb{E}[M_{\infty}M_{\infty}]^{1/2}.
\]

**Proposition 5.1**

The space \(\mathcal{H}^2\) equipped with the scalar product \((M, N)_{\mathcal{H}^2}\) is a Hilbert space.
Clearly \((M, M)_{\mathbb{H}^2} = 0\) if and only \(M = 0\). The scalar product \((M, N)_{\mathbb{H}^2}\) thus yields a norm on \(\mathbb{H}^2\) given by

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\]

**Proposition 5.1**

The space \(\mathbb{H}^2\) equipped with the scalar product \((M, N)_{\mathbb{H}^2}\) is a Hilbert space.
Proof of Proposition 5.1

We need to verify that the vector space $H^2$ is complete for the norm $||M||_{H^2}$. Let $(M^n)_{n \geq 1}$ be a Cauchy sequence in $H^2$. We have then

$$\lim_{m,n \to \infty} E[(M^n_\infty - M^m_\infty)^2] = \lim_{m,n \to \infty} (M^n - M^m, M^n - M^m)_{H^2} = 0.$$ 

Consequently, the sequence $(M^n_\infty)_{n \geq 1}$ converges in $L^2$ to a limit, which we denote by $Z$. On the other hand, Doob's $L^2$-inequality and a straightforward passage to the limit show that, for every $m, n$,

$$E[\sup_{t \geq 0}(M^n_t - M^m_t)^2] \leq 4E[(M^n_\infty - M^m_\infty)^2].$$

Thus

$$\lim_{m,n \to \infty} E[\sup_{t \geq 0}(M^n_t - M^m_t)^2] = 0. \quad (2)$$

Hence, for every $t > 0$, $M^n_t$ converges in $L^2$, and we want to argue that the limit yields a process with continuous sample paths.
Proof of Proposition 5.1 (cont)

To this end, we use (2) to find an increasing sequence $n_k \uparrow \infty$ such that

$$
\mathbb{E} \left[ \sum_{k=1}^{\infty} \sup_{t \geq 0} |M_{t}^{n_k} - M_{t}^{n_{k+1}}| \right] \leq \sum_{k=1}^{\infty} \mathbb{E} \left[ \sup_{t \geq 0} (M_{t}^{n_k} - M_{t}^{n_{k+1}})^2 \right]^{1/2} < \infty.
$$

Thus we have a.s.

$$
\sum_{k=1}^{\infty} \sup_{t \geq 0} |M_{t}^{n_k} - M_{t}^{n_{k+1}}| < \infty.
$$

and thus the sequence $(M_{t}^{n_k})_{t \geq 0}$ converges uniformly on $\mathbb{R}^+$ a.s., to a limit denoted by $(M_{t})_{t \geq 0}$. On the zero probability set where the uniform convergence does not hold, we take $M_{t} = 0$ for every $t > 0$. 

Proof of Proposition 5.1 (cont)

Clearly the limiting process $M$ has continuous sample paths and is adapted. Furthermore, from the $L^2$-convergence of $(M^n_t)_{t \geq 0}$ to $Z$ we immediately get by passing to the limit in the identity $M^n_{tk} = \mathbb{E}[M^n_k | \mathcal{F}_t]$ that $M_t = \mathbb{E}[Z | \mathcal{F}_t]$. Hence $(M_t)_{t \geq 0}$ is a continuous martingale and is bounded in $L^2$, so that $M \in \mathbb{H}^2$. The a.s. uniform convergence of $(M^n_{tk})_{t \geq 0}$ to $(M_t)_{t \geq 0}$ then ensures that $M_\infty = \lim_{k \to \infty} M^n_k = Z$ a.s. Finally, the $L^2$-convergence of $(M^n_{\infty})$ to $Z = M_\infty$ shows that the sequence $(M^n)$ converges to $M$ in $\mathbb{H}^2$. 