Math 562 Fall 2020

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Outline

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HW2 is graded now.

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Recall that in the remainder of this chapter, we assume that $(\mathcal{F}_t)_{t\geq 0}$ is complete.

Theorem 4.9

Let $M = (M_t)_{t\geq 0}$ be a continuous local martingale. There exists an increasing process denoted by $(\langle M, M\rangle_t)_{t\geq 0}$, which is unique up to indistinguishability, such that $M_t^2 - \langle M, M\rangle_t$ is a continuous local martingale. Furthermore, for every fixed $t > 0$, if $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ is an increasing sequence of subdivisions of $[0, t]$ with mesh tending to 0, we have

$$\langle M, M\rangle_t = \lim_{n\to\infty} \sum_{j=0}^{p_n} (M_{t_j^n} - M_{t_{j-1}^n})^2 \tag{1}$$

in probability. The process $\langle M, M\rangle$ is called the quadratic variation of M.

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Proof of Theorem 4.9 (cont)

Last time we proved the theorem in the case when $M_0 = 0$ and M is bounded. This time we will deal with the general case.

Write $M_t = M_0 + N_t$, then $M_t^2 = M_0^2 + 2M_0 N_t + N_t^2$. Since $M_0 N_t$ is a continuous local martingale, we may assume that $M_0 = 0$. We then define

$$T_n = \inf\{t \geq 0 : |M_t| \geq n\}$$

and we can apply the bounded case to the stopped martingales M^{T_n}. Define $A^{[n]} = \langle M^{T_n}, M^{T_n} \rangle$. The uniqueness part of the theorem shows that the processes $(A^{[n+1]}_{t \wedge T_n})$ and $(A^{[n]}_t)$ are indistinguishable. It follows that there exists an increasing process A such that , for every n, $(A_{t \wedge T_n})$ and $(A^{[n]}_t)$ are indistinguishable. By construction, $M^2_{t \wedge T_n} - A_{t \wedge T_n}$ is a martingale for every n, which precisely implies that $M_t^2 - A_t$ is a continuous local martingale. We take $\langle M, M \rangle_t = A_t$, which completes the proof of the existence part of the theorem.

Proof of Theorem 4.9 (cont)

Finally, to get (1), it suffices to consider the case $M_0 = 0$. The bounded case then shows that (1) holds if M and $\langle M, M \rangle$ are replaced respectively by M^{T_n} and $\langle M, M \rangle_{t \wedge T_n}$. Then it is enough to observe that, for every $t > 0$, $\mathbb{P}(t \leq T_n) \to 1$ as $n \to \infty$.

Proposition 4.11

Let M be a continuous local martingale and let T be a stopping time. Then we have a.s. for every $t \geq 0$,

$$\langle M^T, M^T \rangle_t = \langle M, M \rangle_{t \wedge T}.$$

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This follows immediately from the fact that $M^2_{t\wedge T} - \langle M, M\rangle_{t\wedge T}$ is a continuous local martingale.

Proposition 4.12

Let M be a continuous local martingale with $M_0 = 0$. Then $\langle M, M\rangle = 0$ if and only if $M = 0$.

Proof of Proposition 4.12

Suppose $\langle M, M\rangle = 0$. Then M_t^2 is a non-negative continuous local martingale and thus a supermartingale. Hence $\mathbb{E}[M_t^2] \le \mathbb{E}[M_0^2] = 0$, so that $M_t = 0$ for every t. The converse is obvious.

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Theorem 4.13

Let M be a continuous local martingale with $M_0 \in L^2$.

(i) The following are equivalent:

- **(a)** M is a (true) martingale bounded in L^2.
- **(b)** $\mathbb{E}[\langle M, M \rangle_\infty] < \infty$.

Furthermore, if these properties hold, the process $M_t^2 - \langle M, M \rangle_t$ is a uniformly integrable martingale, and in particular, $\mathbb{E}[M_\infty^2] = \mathbb{E}[M_0^2] + \mathbb{E}[\langle M, M \rangle_\infty]$.

(ii) The following are equivalent:

- **(a)** M is a (true) martingale and $M_t \in L^2$ for every $t > 0$.
- **(b)** $\mathbb{E}[\langle M, M \rangle_t] < \infty$ for every $t > 0$.

Furthermore, if these properties hold, the process $M_t^2 - \langle M, M \rangle_t$ is a martingale.

Proof of Theorem 4.13

(i) Replacing M by $M - M_0$, we may assume that $M_0 = 0$. Let us first assume that M is a martingale bounded in L^2. Doob's L^2 inequality shows that, for every $T > 0$,

$$\mathbb{E}[\sup_{0\leq t\leq T} M_t^2] \leq 4\mathbb{E}[M_T^2].$$

Letting $T \uparrow \infty$, we get

$$\mathbb{E}[\sup_{t\geq 0} M_t^2] \leq 4 \sup_{t\geq 0} \mathbb{E}[M_t^2] =: C < \infty.$$

Define $S_n = \inf\{t \geq 0 : \langle M, M\rangle_t \geq n\}$. Then the continuous local martingale $M_{t\wedge S_n}^2 - \langle M, M\rangle_{t\wedge S_n}$ is dominated by

$$\sup_{s\geq 0} M_s^2 + n,$$

which is integrable.

Proof of Theorem 4.13 (cont)

Thus this continuous local martingale is a uniformly integrable martingale, hence

$$\mathbb{E}[\langle M, M\rangle_{t\wedge S_n}] = \mathbb{E}[M^2_{t\wedge S_n}] \le \mathbb{E}[\sup_{s\ge 0} M_s^2] \le C.$$

By letting $n\uparrow\infty$ and then $t\uparrow\infty$ and using MCT, we get $\mathbb{E}[\langle M, M\rangle_\infty] \le C < \infty$.

Conversely, assume $\mathbb{E}[\langle M, M\rangle_\infty] < \infty$. Define $T_n = \inf\{t\ge 0 : |M_t| \ge n\}$. Then the continuous local martingale $M^2_{t\wedge T_n} - \langle M, M\rangle_{t\wedge T_n}$ is dominated by

$$n^2 + \langle M, M\rangle_\infty$$

which is integrable. Thus this continuous local martingale is a uniformly integrable martingale, hence, for every $t\ge 0$,

$$\mathbb{E}[M^2_{t\wedge T_n}] = \mathbb{E}[\langle M, M\rangle_{t\wedge T_n}] \le \mathbb{E}[\langle M, M\rangle_\infty] =: C' < \infty.$$

Proof of Theorem 4.13 (cont)

By letting $n \uparrow \infty$ and using Fatou's lemma, we get $\mathbb{E}[M_t^2] \leq C'$, hence $(M_t)_{t \geq 0}$ is bounded in L^2. The display above shows that $(M_{t \wedge T_n})_{n \geq 1}$ is uniformly integrable, and therefore converges both a.s. and in L^1 to M_t for every $t \geq 0$. Recall that M^{T_n} is a martingale, and so

$$\mathbb{E}[M_{t \wedge T_n} | \mathcal{F}_s] = M_{s \wedge T_n}, \quad 0 \leq s \leq t.$$

The L^1 convergence allows us to take the limit $n \uparrow \infty$ to get

$$\mathbb{E}[M_t | \mathcal{F}_s] = M_s, \quad 0 \leq s \leq t.$$

Finally, if (a) and (b) hold, the continuous local martingale $M^2 - \langle M, M \rangle$ is dominated by the integrable variable

$$\sup_{t \geq 0} M_t^2 + \langle M, M \rangle_\infty$$

and is therefore a uniformly integrable martingale.

(ii) Apply (i) to $(M_{t \wedge a})_{t \geq 0}$ for every $a \geq 0$.

Outline

Definition 4.14

If M and N are two continuous local martingales, the bracket $\langle M, N \rangle$ is the finite variation process defined by setting, for every $t \geq 0$,

$$\langle M, N \rangle_t = \frac{1}{2} \left(\langle M+N, M+N \rangle_t - \langle M, M \rangle_t - \langle N, N \rangle_t \right).$$

Proposition 4.15

(i) $\langle M, N \rangle$ is the unique (up to indistinguishability) finite variation process such that $M_t N_t - \langle M, N \rangle_t$ is a continuous local martingale.

(ii) The map $(M, N) \mapsto \langle M, N \rangle$ is bilinear and symmetric.

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Proposition 4.15 (cont)

(iii) If $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n$ is an increasing sequence of partitions of $[0, t]$ with mesh tending to 0,

$$\lim_{n\to\infty} \sum_{j=1}^{p_n} (M_{t_j^n} - M_{t_{j-1}^n})(N_{t_j^n} - N_{t_{j-1}^n}) = \langle M, N \rangle_t$$

in probability.

(iv) For every stopping time T, $\langle M^T, N^T \rangle_t = \langle M^T, N \rangle_t = \langle M, N \rangle_{t\wedge T}$.

(v) If M and N are two martingales (with continuous sample paths) bounded in L^2, then $M_t N_t - \langle M, N \rangle_t$ is a uniformly integrable martingale. Consequently $\langle M, N \rangle_\infty$ is well defined as the almost sure limit of $\langle M, N \rangle_t$ as $t \to \infty$, is integrable, and satisfies

$$\mathbb{E}[M_\infty N_\infty] = \mathbb{E}[M_0 N_0] + \mathbb{E}[\langle M, N \rangle_\infty].$$

Proof of Proposition 4.15

(i) This follows from the analogous characterization in Theorem 4.9.

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(ii) Follows from (iii)

(iv) (iii) implies, for $0 \le s \le t$, a.s.

$$\langle M^T, N^T\rangle_t = \langle M^T, N\rangle_t = \langle M, N\rangle_t \qquad \text{on } \{T \ge t\}$$
$$\langle M^T, N^T\rangle_t - \langle M^T, N^T\rangle_s = \langle M^T, N\rangle_t - \langle M^T, N\rangle_s = 0 \text{ on } \{T \le s < t\}.$$

(v) Thus is a consequence of Theorem 4.13 (i).

Proposition 4.16

Let B and B' be two independent $(\mathcal{F}_t)$-Brownian motions. Then $\langle B, B' \rangle_t = 0$ for every $t \geq 0$.

Proof of Proposition 4.16

By subtracting the initial values, we may assume that $B_0 = B'_0 = 0$. We then observe that the process $X_t = \frac{1}{\sqrt{2}}(B_t + B'_t)$ is a martingale, as a linear combination of martingales. By checking the finite-dimensional marginals of X_t, we verify that X is also a Brownian motion. Proposition 2.16 implies that $\langle X, X \rangle_t = t$, and, using the bilinearity of the bracket, it follows that $\langle B, B' \rangle_t = 0$.

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Definition 4.17

Two continuous local martingales M and N are said to be orthogonal if $\langle M, N \rangle = 0$, which holds if and only if MN is a continuous local martingale.

In particular, two independent $(\mathcal{F}_t)$-Brownian motions are orthogonal martingales, by Proposition 4.16.

If M and N are two orthogonal martingales bounded in L^2, we have $\mathbb{E}[M_t N_t] = \mathbb{E}[M_0 N_0]$, and even $\mathbb{E}[M_S N_S] = \mathbb{E}[M_0 N_0]$ for every stopping time S.

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Proposition 4.18 (Kunita-Watanabe)

Let M and N be two continuous local martingales and let H and K be two measurable processes. Then, a.s.,

$$\int_0^\infty |H_s||K_s||d\langle M, N\rangle_s| \leq \left(\int_0^\infty H_s^2 d\langle M, M\rangle_s\right)^{1/2} \left(\int_0^\infty K_s^2 d\langle N, N\rangle_s\right)^{1/2}$$