Outline
Outline

1. General Info
2. 4.2 Continuous Local Martingales
3. 4.3 The Quadratic Variation of a Continuous Local Martingale
HW3 is due on Friday, 10/02, at noon. Please submit your HW3 via the course Moodle page. Make sure that your HW is uploaded successfully.

HW2 is graded now.
HW3 is due on Friday, 10/02, at noon. Please submit your HW3 via the course Moodle page. Make sure that your HW is uploaded successfully.

HW2 is graded now.
Outline

1. General Info
2. 4.2 Continuous Local Martingales
3. 4.3 The Quadratic Variation of a Continuous Local Martingale
Properties of continuous local martingales

(a) A martingale \( M \) with continuous sample paths is a continuous local martingale, and the sequence \( T_n = n \) reduces \( M \).

(b) In the definition of a continuous local martingale starting from 0, one can replace “uniformly integrable martingale” by “martingale” (indeed, one can then observe that \( M^{T_n \wedge n} \) is uniformly integrable, and we still have \( T_n \wedge n \uparrow \infty \).

(c) If \( M \) is a continuous local martingale, then, for every stopping time \( T \), \( M^T \) is a continuous local martingale

(d) If \( (T_n) \) reduces \( M \) and if \( (S_n) \) is a sequence of stopping times such that \( S_n \uparrow \infty \), then the sequence \( (T_n \wedge S_n) \) also reduces \( M \).

(e) The space of all continuous local martingales is a vector space (If \( M \) and \( M' \) are two continuous local martingales such that \( M_0 = M'_0 = 0 \), if \( (T_n) \) reduces \( M \) and \( (T'_n) \) reduces \( M' \), then \( (T_n \wedge T'_n) \) reduces \( M + M' \)).
Proposition 4.7

(i) A non-negative continuous local martingale $M$ such that $M_0 \in L^1$ is a supermartingale.

(ii) A continuous local martingale $M$ such that there exists a random variable $Z \in L^1$ such that $|M_t| \leq Z$ for every $t \geq 0$ is a uniformly integrable martingale.

(iii) If $M$ is a continuous local martingale and $M_0 = 0$ (or more generally $M_0 \in L^1$), the sequence of stopping times

$$T_n = \inf\{t \geq 0 : |M_t| \geq n\}$$

reduces $M$. 
(i) Write $M_t = M_0 + N_t$. By definition, there exists a sequence $(T_n)$ of stopping times that reduces $N$. Then, if $s \leq t$, we have for every $n$, $N_{s \wedge T_n} = \mathbb{E}[N_{t \wedge T_n} | \mathcal{F}_s]$ which implies by adding $M_0$ (using $M_0 \in L^1$) that $M_{s \wedge T_n} = \mathbb{E}[M_{t \wedge T_n} | \mathcal{F}_s]$.

Since $M$ is non-negative, we can let $n \uparrow \infty$ and apply the version of Fatou’s lemma for conditional expectations to get $M_s \geq \mathbb{E}[M_t | \mathcal{F}_s]$.

Taking $s = 0$, we get $\mathbb{E}[M_t] \leq \mathbb{E}[M_0] < \infty$, hence $M_t \in L^1$ for every $t \geq 0$. Thus $M$ is a supermartingale.
Proof of Proposition 4.7 (cont)

(ii) By the same argument as in (i), we get for $0 \leq s \leq t$,

$$M_{s \wedge T_n} = \mathbb{E}[M_{t \wedge T_n} \mid \mathcal{F}_s].$$

Since $|M_{t \wedge T_n}| \leq Z$, we can apply DCT to get that $M_{t \wedge T_n} \to M_t$ in $L^1$. Thus $M_s = \mathbb{E}[M_t \mid \mathcal{F}_s]$.

(iii) Suppose that $M_0 = 0$. $T_n$ is a stopping time for each $n$. The desired result is an immediate consequence of (ii) since $M_{T_n}^T$ is a continuous local martingale and $|M_{T_n}^T| \leq n$. If we only assume that $M_0 \in L^1$, then $|M_{T_n}^T| \leq n + |M_0|$.

A continuous local martingale $M$ such that $(M_t)_{t \geq 0}$ is uniformly integrable may not be a martingale.
Proof of Proposition 4.7 (cont)

(ii) By the same argument as in (i), we get for $0 \leq s \leq t$,

$$M_{s \wedge T_n} = \mathbb{E}[M_{t \wedge T_n} | \mathcal{F}_s].$$

Since $|M_{t \wedge T_n}| \leq Z$, we can apply DCT to get that $M_{t \wedge T_n} \to M_t$ in $L^1$. Thus $M_s = \mathbb{E}[M_t | \mathcal{F}_s]$.

(iii) Suppose that $M_0 = 0$. $T_n$ is a stopping time for each $n$. The desired result is an immediate consequence of (ii) since $M_{T_n}$ is a continuous local martingale and $|M_{t \wedge T_n}| \leq n$. If we only assume that $M_0 \in L^1$, then $|M_{t \wedge T_n}| \leq n + |M_0|$.

A continuous local martingale $M$ such that $(M_t)_{t \geq 0}$ is uniformly integrable may not be a martingale.
**Theorem 4.8**

Let $M$ be a continuous local martingale. Assume that $M$ is also a finite variation process (in particular $M_0 = 0$). Then $M_t = 0$ for every $t \geq 0$.

**Proof of Theorem 4.8**

Define, for $n \geq 0$

$$\tau_n = \inf\{t \geq 0 : \int_0^t |dM_s| \geq n\}.$$ 

The, for each $n \geq 0$, $\tau_n$ is a stopping time.

Fix $n \geq 0$ and let $N = M^{\tau_n}$. Note that, for every $t \geq 0$,

$$|N_t| = |M_{t \wedge \tau_n}| \leq \int_0^{t \wedge \tau_n} |dM_s| \leq n.$$ 

So by Proposition 4.7 $N$ is a bounded martingale. Let $t > 0$ and let $0 = t_0 < t_1 < \cdots < t_p = t$ be a partition of $[0, t]$. 


Theorem 4.8
Let $M$ be a continuous local martingale. Assume that $M$ is also a finite variation process (in particular $M_0 = 0$). Then $M_t = 0$ for every $t \geq 0$.

Proof of Theorem 4.8
Define, for $n \geq 0$

$$\tau_n = \inf \{ t \geq 0 : \int_0^t |dM_s| \geq n \}.$$ 

The, for each $n \geq 0$, $\tau_n$ is a stopping time.
Fix $n \geq 0$ and let $N = M_{\tau_n}$. Note that, for every $t \geq 0$,

$$|N_t| = |M_{t \wedge \tau_n}| \leq \int_0^{t \wedge \tau_n} |dM_s| \leq n.$$ 

So by Proposition 4.7 $N$ is a bounded martingale. Let $t > 0$ and let $0 = t_0 < t_1 < \cdots < t_p = t$ be a partition of $[0, t]$. 

Proof of Theorem 4.8 (cont)

Then, from Proposition 3.14, we have

\[ \mathbb{E}[N_t^2] = \sum_{j=1}^{p} \mathbb{E}[(N_{t_j} - N_{t_{j-1}})^2] \]

\[ \leq \mathbb{E} \left[ \left( \sup_{1 \leq j \leq p} |N_{t_j} - N_{t_{j-1}}| \right) \sum_{j=1}^{p} |N_{t_j} - N_{t_{j-1}}| \right] \leq n \mathbb{E} \left[ \sup_{1 \leq j \leq p} |N_{t_j} - N_{t_{j-1}}| \right]. \]

Apply the preceding bound to a sequence \( 0 = t^k_0 < t^k_1 < \cdots < t^k_p = t \) of partitions of \([0, t]\) whose mesh tends to 0. Using the continuity of sample paths and the fact that \( N \) is bounded, we get

\[ \lim_{k \to \infty} \mathbb{E} \left[ \sup_{1 \leq j \leq p} |N_{t^k_j} - N_{t^k_{j-1}}| \right] = 0. \]

Thus \( \mathbb{E}[N_t^2] = 0 \), hence \( M_t \wedge \tau_n = 0 \) a.s. Letting \( n \uparrow \infty \), we get \( M_t = 0 \) a.s.
Outline

1. General Info

2. 4.2 Continuous Local Martingales

3. 4.3 The Quadratic Variation of a Continuous Local Martingale
In the remainder of this chapter, we assume that \((\mathcal{F}_t)_{t \geq 0}\) is complete.

**Theorem 4.9**

Let \(M = (M_t)_{t \geq 0}\) be a continuous local martingale. There exists an increasing process denoted by \((\langle M, M \rangle_t)_{t \geq 0}\), which is unique up to indistinguishability, such that \(M_t^2 - \langle M, M \rangle_t\) is a continuous local martingale. Furthermore, for every fixed \(t > 0\), if \(0 = t^n_0 < t^n_1 < \cdots < t^n_{p_n} = t\) is an increasing sequence of subdivisions of \([0, t]\) with mesh tending to 0, we have

\[
\langle M, M \rangle_t = \lim_{n \to \infty} \sum_{j=0}^{p_n} (M_{t^n_j} - M_{t^n_{j-1}})^2
\]  

in probability. The process \(\langle M, M \rangle\) is called the quadratic variation of \(M\).
In the remainder of this chapter, we assume that \((\mathcal{F}_t)_{t \geq 0}\) is complete.

**Theorem 4.9**

Let \(M = (M_t)_{t \geq 0}\) be a continuous local martingale. There exists an increasing process denoted by \((\langle M, M \rangle_t)_{t \geq 0}\), which is unique up to indistinguishability, such that \(M_t^2 - \langle M, M \rangle_t\) is a continuous local martingale. Furthermore, for every fixed \(t > 0\), if \(0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t\) is an increasing sequence of subdivisions of \([0, t]\) with mesh tending to 0, we have

\[
\langle M, M \rangle_t = \lim_{n \to \infty} \sum_{j=0}^{p_n} (M_{t_j^n} - M_{t_{j-1}^n})^2
\]

in probability. The process \(\langle M, M \rangle\) is called the quadratic variation of \(M\).
If $M = B$ is an $(\mathcal{F}_t)$-Brownian motion, then $B$ is a continuous martingale with continuous sample paths. Since $B^2_t - t$ is a martingale, $\langle B, B \rangle_t = t$.

**Remark**

The process $\langle M, M \rangle$ does not depend on the initial value $M_0$, but only on the increments of $M$: if $M_t = M_0 + N_t$, then $\langle M, M \rangle = \langle N, N \rangle$. 
If $M = B$ is an $(\mathcal{F}_t)$-Brownian motion, then $B$ is a continuous martingale with continuous sample paths. Since $B_t^2 - t$ is a martingale, $\langle B, B \rangle_t = t$.

**Remark**

The process $\langle M, M \rangle$ does not depend on the initial value $M_0$, but only on the increments of $M$: if $M_t = M_0 + N_t$, then $\langle M, M \rangle = \langle N, N \rangle$. 
Proof of Theorem 4.9

We start by proving the first assertion. Uniqueness is an easy consequence of Theorem 4.8. Indeed, let $A$ and $A'$ be two increasing processes satisfying the condition given in the statement. Then the process $A_t - A'_t = (M^2_t - A'_t) - (M^2 - A_t)$ is both a continuous local martingale and a finite variation process. It follows that $A - A' = 0$.

To prove existence, consider first the case where $M_0 = 0$ and $M$ is bounded (and thus a martingale). Fix $K > 0$ and an increasing sequence $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = K$ of partitions of $[0, K]$ with mesh tending to 0.

Note that, for all $0 \leq r < s$ and every bounded $\mathcal{F}_r$-measurable random variable $Z$, the process

$$N_t = Z(M_{s \wedge t} - M_{r \wedge t})$$

is a martingale.
Proof of Theorem 4.9 (cont)

It follows that, for every \( n \), the process

\[
X_t^n = \sum_{j=1}^{p_n} M_{t_{j-1}^n} (M_{t_{j}^n} \wedge t_{j-1}^n - M_{t_{j-1}^n} \wedge t_{j-1}^n)
\]

is a (bounded) martingale. The reason for considering these martingales comes from the following identity, which results from a simple calculation: for every \( n \), for every \( j \in \{1, \ldots, p_n\} \),

\[
M_{t_{j}^n}^2 - 2X_{t_{j}^n}^n = \sum_{i=1}^{j} (M_{t_{i}^n}^n - M_{t_{i-1}^n}^n)^2. \tag{2}
\]

Lemma 4.10

It holds that

\[
\lim_{n,m \to \infty} \mathbb{E}[(X_K^n - X_K^m)^2] = 0.
\]
Proof of Theorem 4.9 (cont)

It follows that, for every \( n \), the process

\[
X_t^n = \sum_{j=1}^{p_n} M_{t^n_{j-1}} (M_{t^n_{j-1}} - M_{t^n_{j-1}} - 1)
\]

is a (bounded) martingale. The reason for considering these martingales comes from the following identity, which results from a simple calculation: for every \( n \), for every \( j \in \{1, \ldots, p_n\} \),

\[
M_{t^n_{j}}^2 - 2X_{t^n_{j}} = \sum_{i=1}^{j} (M_{t^n_{i}} - M_{t^n_{i-1}})^2.
\]  \(2\)

Lemma 4.10

It holds that

\[
\lim_{n,m \to \infty} \mathbb{E}[(X_K^n - X_K^m)^2] = 0.
\]
Proof of Theorem 4.9 (cont)

By Doob's inequality and lemma 4.10, we have

$$\lim_{n,m \to \infty} \mathbb{E}[\sup_{t \leq K} (X^n_t - X^m_t)^2] = 0.$$ 

In particular, for every $t \in [0, K]$, $(X^n_t)_{n \geq 0}$ is a Cauchy sequence in $L^2$ and thus converges in $L^2$. We want to argue that the limit yields a process $Y$ indexed by $[0, K]$ with continuous sample paths. To see this, we note that the display above allows us find a strictly increasing sequence $(n_k)_{k \geq 1}$ of positive integers such that, for every $k \geq 1$,

$$\mathbb{E} \left[ \sup_{t \leq K} (X^{n_k+1}_t - X^{n_k}_t)^2 \right] \leq 2^{-k}.$$

This implies that

$$\mathbb{E} \left[ \sum_{k=1}^{\infty} \sup_{t \leq K} |X^{n_k+1}_t - X^{n_k}_t| \right] < \infty.$$
Proof of Theorem 4.9 (cont)

and thus

$$\sum_{k=1}^{\infty} \sup_{t \leq K} |X_{t}^{n_k+1} - X_{t}^{n_k}| < \infty, \text{ a.s.}$$

Consequently, except on the negligible set $\mathcal{N}$ where the series in the last display diverges, the sequence of random functions $(X_{t}^{n_k}: 0 \leq t \leq K)$ converges uniformly on $[0, K]$ as $k \to \infty$, and the limiting random function is continuous by uniform convergence. We can thus define $Y_t(\omega) = \lim_{k \to \infty} X_{t}^{n_k}(\omega)$ for every $t \in [0, K]$, if $\omega \in \mathcal{N}^c$, and $Y_t(\omega) = 0$ for every $t \in [0, K]$, if $\omega \in \mathcal{N}$. The process $(Y_t)_{0 \leq t \leq K}$ has continuous sample paths and $Y_t \in \mathcal{F}_t$ by completeness.

Furthermore, since the $L^2$ limit of $(X_{t}^n)_{n \geq 1}$ must coincide with the a.s. limit of a subsequence, $Y_t$ is also the limit of $X_{t}^n$ in $L^2$, for every $t \in [0, K]$, and we can pass to the limit in the martingale property for $X^n$, to obtain $\mathbb{E}[Y_t | \mathcal{F}_s] = Y_s$ for every $0 \leq s \leq t \leq K$. Thus $(Y_{t \land K})_{t \geq 0}$ is a martingale with continuous sample paths.
Proof of Theorem 4.9 (cont)

On the other hand, the identity (2) shows that the sample paths of the process \( M^2_t - 2X^n_t \) are non-decreasing along the finite sequence \((t^n_j : 1 \leq j \leq p_n)\). By passing to the limit \( k \to \infty \) along the sequence \((n_k)_{k \geq 1}\), we get that the sample paths of \( M^2_t - 2Y_t \) are non-decreasing on \([0, K]\), except maybe on the negligible set \( \mathcal{N} \). For every \( t \in [0, K] \), define \( A_t^{(K)} = M^2_t - 2Y_t \) on \( \mathcal{N}^c \) and \( Y_t^{(K)} = 0 \) on \( \mathcal{N} \). Then \( A_0^{(K)} = 0, A_t^{(K)} \in \mathcal{F}_t \), \( A^{(K)} \) has non-decreasing continuous sample paths, and \((M^2_{t \wedge K} - A_{t \wedge K})_{t \geq 0}\) is a martingale.

We apply the preceding considerations with \( K = l \), for every integer \( l \geq 1 \), and we get a process \((A^{(l)}_t)_{0 \leq t \leq l}\). Observe that, for every integer \( l \geq 1 \), \( A^{(l+1)}_t = A^{(l)}_t \) for every \( t \geq 0 \) a.s. by the uniqueness argument explained at the beginning of the proof. It follows that we can define an increasing process \( \langle M, M \rangle \) such that \( \langle M, M \rangle_t = A^{(l)}_t \) for every \( t \in [0, l] \) and every \( l \geq 1 \) a.s. and clearly \( M^2_t - \langle M, M \rangle_t \) is a martingale.
Proof of Theorem 4.9 (cont)

In order to get (1), we observe that, if $K > 0$ and the sequence of partitions $0 = t^n_0 < t^n_1 < \cdots < t^n_{p_n} = K$ are fixed, the process $A^{(K)}_{t \land K}$ must be indistinguishable from $\langle M, M \rangle_{t \land K}$, again by the uniqueness argument (we know that both $M^2_{t \land K} - A^{(K)}_{t \land K}$ and $M^2_{t \land K} - \langle M, M \rangle_{t \land K}$ are martingales). In particular, we have $\langle M, M \rangle_K = A^{(K)}_K$ a.s. Then, from (2), with $j = p_n$, and the fact that $X^n_K$ converges in $L^2$ to $Y_K = \frac{1}{2} (M^2_K - A^{(K)}_K)$ we get that

$$\lim_{n \to \infty} \sum_{j=1}^{p_n} (M^n_{t_j} - M^n_{t_{j-1}})^2 = \langle M, M \rangle_K \quad \text{in } L^2.$$  

This completes the proof of the theorem in the case when $M_0 = 0$ and $M$ is bounded.