Math 562 Fall 2020

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Outline

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HW3 is due on Friday, 10/02, at noon. Please submit your HW3 via the course Moodle page. Make sure that your HW is uploaded successfully.

HW2 is graded now.

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Properties of continuous local martingales

(a) A martingale M with continuous sample paths is a continuous local martingale, and the sequence $T_n = n$ reduces M.

(b) In the definition of a continuous local martingale starting from 0, one can replace "uniformly integrable martingale" by "martingale" (indeed, one can then observe that $M^{T_n \wedge n}$ is uniformly integrable, and we still have $T_n \wedge n \uparrow \infty$.

(c) If M is a continuous local martingale, then, for every stopping time T, M^T is a continuous local martingale

(d) If (T_n) reduces M and if (S_n) is a sequence of stopping times such that $S_n \uparrow \infty$, then the sequence $(T_n \wedge S_n)$ also reduces M.

(e) The space of all continuous local martingales is a vector space (If M and M' are two continuous local martingales such that $M_0 = M'_0 = 0$, if (T_n) reduces M and (T'_n) reduces M', then $(T_n \wedge T'_n)$ reduces $M + M'$.).

Proposition 4.7

(i) A non-negative continuous local martingale M such that $M_0 \in L^1$ is a supermartingale.

(ii) A continuous local martingale M such that there exists a random variable $Z \in L^1$ such that $|M_t| \leq Z$ for every $t \geq 0$ is a uniformly integrable martingale.

(iii) If M is a continuous local martingale and $M_0 = 0$ (or more generally $M_0 \in L^1$), the sequence of stopping times

$$T_n = \inf\{t \geq 0 : |M_t| \geq n\}$$

reduces M.

Proof of Proposition 4.7

(i) Write $M_t = M_0 + N_t$. By definition, there exists a sequence (T_n) of stopping times that reduces N. Then, if $s \le t$, we have for every n,

$$N_{s\wedge T_n} = \mathbb{E}[N_{t\wedge T_n}|\mathcal{F}_s]$$

which implies by adding M_0 (using $M_0 \in L^1$) that

$$M_{s\wedge T_n} = \mathbb{E}[M_{t\wedge T_n}|\mathcal{F}_s].$$

Since M is non-negative, we can let $n \uparrow \infty$ and apply the version of Fatou's lemma for conditional expectations to get

$$M_s \ge \mathbb{E}[M_t|\mathcal{F}_s].$$

Taking $s = 0$, we get $\mathbb{E}[M_t] \le \mathbb{E}[M_0] < \infty$, hence $M_t \in L^1$ for every $t \ge 0$. Thus M is a supermartingale.

Proof of Proposition 4.7 (cont)

(ii) By the same argument as in (i), we get for $0 \leq s \leq t$,

$$M_{s\wedge T_n} = \mathbb{E}[M_{t\wedge T_n}|\mathcal{F}_s].$$

Since $|M_{t\wedge T_n}| \leq Z$, we can apply DCT to get that $M_{t\wedge T_n} \to M_t$ in L^1. Thus $M_s = \mathbb{E}[M_t|\mathcal{F}_s]$.

(iii) Suppose that $M_0 = 0$. T_n is a stopping time for each n. The desired result is an immediate consequence of (ii) since M^{T_n} is a continuous local martingale and $|M_t^{T_n}| \leq n$. If we only assume that $M_0 \in L^1$, then $|M_t^{T_n}| \leq n + |M_0|$.

A continuous local martingale M such that $(M_t)_{t\geq 0}$ is uniformly integrable may not be a martingale.

Proof of Proposition 4.7 (cont)

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A continuous local martingale M such that $(M_t)_{t \geq 0}$ is uniformly integrable may not be a martingale.

Theorem 4.8

Let M be a continuous local martingale. Assume that M is also a finite variation process (in particular $M_0 = 0$). Then $M_t = 0$ for every $t \geq 0$.

Proof of Theorem 4.8

Define, for $n \geq 0$

$$\tau_n = \inf\{t \geq 0 : \int_0^t |dM_s| \geq n\}.$$

The, for each $n \geq 0$, τ_n is a stopping time.
Fix $n \geq 0$ and let $N = M^{\tau_n}$. Note that, for every $t \geq 0$,

$$|N_t| = |M_{t\wedge\tau_n}| \leq \int_0^{t\wedge\tau_n} |dM_s| \leq n.$$

So by Proposition 4.7 N is a bounded martingale. Let $t > 0$ and let $0 = t_0 < t_1 < \cdots < t_p = t$ be a partition of $[0, t]$.

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Proof of Theorem 4.8 (cont)

Then, from Proposition 3.14, we have

$$\mathbb{E}[N_t^2] = \sum_{j=1}^{p} \mathbb{E}[(N_{t_j} - N_{t_{j-1}})^2]$$

$$\leq \mathbb{E}\left[\left(\sup_{1\leq j\leq p} |N_{t_j} - N_{t_{j-1}}|\right) \sum_{j=1}^{p} |N_{t_j} - N_{t_{j-1}}|\right] \leq n\mathbb{E}\left[\sup_{1\leq j\leq p} |N_{t_j} - N_{t_{j-1}}|\right].$$

Apply the preceding bound to a sequence $0 = t_0^k < t_1^k < \cdots < t_{p_k}^k = t$ of partitions of $[0, t]$ whose mesh tends to 0. Using the continuity of sample paths and the fact that N is bounded, we get

$$\lim_{k\to\infty} \mathbb{E}\left[\sup_{1\leq j\leq p_k} |N_{t_j^k} - N_{t_{j-1}^k}|\right] = 0.$$

Thus $\mathbb{E}[N_t^2] = 0$, hence $M_{t\wedge\tau_n} = 0$ a.s. Letting $n \uparrow \infty$, we get $M_t = 0$ a.s.

Outline

In the remainder of this chapter, we assume that $(\mathcal{F}_t)_{t\geq 0}$ is complete.

Theorem 4.9

Let $M = (M_t)_{t\geq 0}$ be a continuous local martingale. There exists an increasing process denoted by $(\langle M, M\rangle_t)_{t\geq 0}$, which is unique up to indistinguishability, such that $M_t^2 - \langle M, M\rangle_t$ is a continuous local martingale. Furthermore, for every fixed $t > 0$, if $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ is an increasing sequence of subdivisions of $[0, t]$ with mesh tending to 0, we have

$$\langle M, M\rangle_t = \lim_{n\to\infty} \sum_{j=0}^{p_n} (M_{t_j^n} - M_{t_{j-1}^n})^2 \qquad (1)$$

in probability. The process $\langle M, M\rangle$ is called the quadratic variation of M.

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If $M = B$ is an $(\mathcal{F}_t)$-Brownian motion, then B is a continuous martingale with continuous sample paths. Since $B_t^2 - t$ is a martingale, $\langle B, B \rangle_t = t$.

Remark

The process $\langle M, M \rangle$ does not depend on the initial value M_0, but only on the increments of M: if $M_t = M_0 + N_t$, then $\langle M, M \rangle = \langle N, N \rangle$.

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Proof of Theorem 4.9

We start by proving the first assertion. Uniqueness is an easy consequence of Theorem 4.8. Indeed, let A and A' be two increasing processes satisfying the condition given in the statement. Then the process $A_t - A'_t = (M_t^2 - A'_t) - (M^2 - A_t)$ is both a continuous local martingale and a finite variation process. It follows that $A - A' = 0$.

To prove existence, consider first the case where $M_0 = 0$ and M is bounded (and thus a martingale). Fix $K > 0$ and an increasing sequence $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = K$ of partitions of $[0, K]$ with mesh tending to 0.

Note that, for all $0 \leq r < s$ and every bounded $\mathcal{F}_r$-measurable random variable Z, the process

$$N_t = Z(M_{s\wedge t} - M_{r\wedge t})$$

is a martingale.

Proof of Theorem 4.9 (cont)

It follows that, for every n, the process

$$X_t^n = \sum_{j=1}^{p_n} M_{t_{j-1}^n}(M_{t \wedge t_j^n} - M_{t \wedge t_{j-1}^n})$$

is a (bounded) martingale. The reason for considering these martingales comes from the following identity, which results from a simple calculation: for every n, for every $j \in \{1, \dots, p_n\}$,

$$M_{t_j^n}^2 - 2X_{t_j^n}^n = \sum_{i=1}^{j} (M_{t_i^n} - M_{t_{i-1}^n})^2. \qquad (2)$$

Lemma 4.10

It holds that

$$\lim_{n,m \to \infty} \mathbb{E}[(X_K^n - X_K^m)^2] = 0.$$

Proof of Theorem 4.9 (cont)

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Proof of Theorem 4.9 (cont)

By Doob's inequality and lemma 4.10, we have

$$\lim_{n,m\to\infty} \mathbb{E}[\sup_{t\le K}(X_t^n - X_t^m)^2] = 0.$$

In particular, for every $t \in [0, K]$, $(X_t^n)_{n\ge 0}$ is a Cauchy sequence in L^2 and thus converges in L^2. We want to argue that the limit yields a process Y indexed by $[0, K]$ with continuous sample paths. To see this, we note that the display above allows us find a strictly increasing sequence $(n_k)_{k\ge 1}$ of positive integers such that, for every $k \ge 1$,

$$\mathbb{E}\left[\sup_{t\le K}(X_t^{n_k+1} - X_t^{n_k})^2\right] \le 2^{-k}.$$

This implies that

$$\mathbb{E}\left[\sum_{k=1}^{\infty}\sup_{t\le K}|X_t^{n_k+1} - X_t^{n_k}|\right] < \infty,$$

Proof of Theorem 4.9 (cont)

and thus

$$\sum_{k=1}^{\infty} \sup_{t \le K} |X_t^{n_k+1} - X_t^{n_k}| < \infty, a.s.$$

Consequently, except on the negligible set $\mathcal{N}$ where the series in the last display diverges, the sequence of random functions $(X_t^{n_k} : 0 \le t \le K)$ converges uniformly on $[0, K]$ as $k \to \infty$, and the limiting random function is continuous by uniform convergence. We can thus define $Y_t(\omega) = \lim_{k\to\infty} X_t^{n_k}(\omega)$ for every $t \in [o, K]$, if $\omega \in \mathcal{N}^c$, and $Y_t(\omega) = 0$ for every $t \in [o, K]$, if $\omega \in \mathcal{N}$. The process $(Y_t)_{0 \le t \le K}$ has continuous sample paths and $Y_t \in \mathcal{F}_t$ by completeness. Furthermore, since the L^2 limit of $(X_t^n)_{n\ge1}$ must coincide with the a.s. limit of a subsequence, Y_t is also the limit of X_t^n in L^2, for every $t \in [0, K]$, and we can pass to the limit in the martingale property for X^n, to obtain $\mathbb{E}[Y_t | \mathcal{F}_s] = Y_s$ for every $0 \le s \le t \le K$. Thus $(Y_{t\wedge K})_{t\ge0}$ is a martingale with continuous sample paths.

Proof of Theorem 4.9 (cont)

On the other hand, the identity (2) shows that the sample paths of the process $M_t^2 - 2X_t^n$ are non-decreasing along the finite sequence $(t_j^n : 1 \leq j \leq p_n)$. By passing to the limit $k \to \infty$ along the sequence $(n_k)_{k\geq 1}$, we get that the sample paths of $M_t^2 - 2Y_t$ are non-decreasing on $[0, K]$, except maybe on the negligible set $\mathcal{N}$. For every $t \in [0, K]$, define $A_t^{(K)} = M_t^2 - 2Y_t$ on $\mathcal{N}^c$ and $Y_t^{(K)} = 0$ on $\mathcal{N}$. Then $A_0^{(K)} = 0$, $A_t^{(K)} \in \mathcal{F}_t$, $A^{(K)}$ has non-decreasing continuous sample paths, and $(M_{t\wedge K}^2 - A_{t\wedge K}^{(K)})_{t\geq 0}$ is a martingale.

We apply the preceding considerations with $K = l$, for every integer $l \geq 1$, and we get a process $(A_t^{(l)})_{0\leq t\leq l}$. Observe that, for every integer $l \geq 1$, $A_{t\wedge l}^{(l+1)} = A_{t\wedge l}^{(l)}$ for every $t \geq 0$ a.s. by the uniqueness argument explained at the beginning of the proof. It follows that we can define an increasing process $\langle M, M\rangle$ such that $\langle M, M\rangle_t = A_t^{(l)}$ for every $t \in [0, l]$ and every $l \geq 1$ a.s. and clearly $M_t^2 - \langle M, M\rangle_t$ is a martingale.

Proof of Theorem 4.9 (cont)

In order to get (1), we observe that, if $K > 0$ and the sequence of partitions $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = K$ are fixed, the process $A_{t\wedge K}^{(K)}$ must be indistinguishable from $\langle M, M\rangle_{t\wedge K}$, again by the uniqueness argument (we know that both $M_{t\wedge K}^2 - A_{t\wedge K}^{(K)}$ and $M_{t\wedge K}^2 - \langle M, M\rangle_{t\wedge K}$ are martingales). In particular, we have $\langle M, M\rangle_K = A_K^{(K)}$ a.s. Then, from (2), with $j = p_n$, and the fact that X_K^n converges in L^2 to $Y_K = \frac{1}{2}(M_K^2 - A_K^{(K)})$ we get that

$$\lim_{n\to\infty} \sum_{j=1}^{p_n} (M_{t_j^n} - M_{t_{j-1}^n})^2 = \langle M, M\rangle_K \quad \text{in } L^2.$$

This completes the proof of the theorem in the case when $M_0 = 0$ and M is bounded.