Math 562 Fall 2020

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Outline

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HW3 is due on Friday, 10/02, at noon. Please submit your HW3 via the course Moodle page. Make sure that your HW is uploaded successfully.

HW2 is graded now.

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Proposition 4.2

For every $t \in [0, T]$,

$$\int_0^t |da(s)| = \sup\{\sum_{j=1}^{p} |a(t_j) - a(t_{j-1})|\},$$

where the supremum is over all partitions $0 = t_0 < t_1 < \cdots < t_p = t$ of $[0, t]$. More precisely, for any increasing sequence $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ of partitions of $[0, t]$ whose mesh tends to 0, we have

$$\lim_{n\to\infty} \sum_{j=1}^{p_n} |a(t_j^n) - a(t_{j-1}^n)| = \int_0^t |da(s)|.$$

In the usual presentation of functions with finite variation, one starts from the property that the supremum in the first display of the proposition is finite.

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Proof of Proposition 4.2

It suffices to treat the case $t = T$. For any subdivision $0 = t_0 < t_1 < \cdots < t_p = T$ of $[0, T]$,

$$|a(t_j) - a(t_{j-1})| = |\mu((t_{j-1}, t_j])| \le |\mu|((t_{j-1}, t_j])$$

and

$$\sum_{j=1}^{p} |\mu|((t_{j-1}, t_j]) = |\mu|([0, T]) = \int_0^T |da(s)|.$$

Thus

$$\int_0^t |da(s)| \ge \sup\{\sum_{j=1}^{p} |a(t_j) - a(t_{j-1})|\}.$$

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Proof of Proposition 4.2 (cont)

Consider an increasing sequence $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = T$ of partitions of $[0, T]$ whose mesh $\max\{t_j^n - t_{j-1}^n : 1 \leq j \leq p_n\}$ tends to 0. Although we are proving a "deterministic" result, we will use a martingale argument. Leaving aside the trivial case where $|\mu| = 0$, we introduce the probability space $\Omega = [0, T]$, equipped with the Borel σ-field $\mathcal{B} = \mathcal{B}([0, T])$ and the probability measure $\mathbb{P}(ds) = (|\mu|([0, T]))^{-1}|\mu|(ds)$. On this probability space, we consider the discrete filtration $(\mathcal{B}_n)_{n\geq 0}$ such that for every $n \geq 0$, $\mathcal{B}_n$ is the σ-field generated by the intervals $(t_{j-1}^n, t_j^n]$, $1 \leq j \leq p_n$. We then define

$$X(s) = 1_{D_+} - 1_{D_-} = \frac{d\mu}{d|\mu|}(s)$$

and

$$X_n = \mathbb{E}[X|\mathcal{B}_n].$$

Proof of Proposition 4.2 (cont)

X_n must be constant on every interval $(t^n_{j-1}, t^n_j]$ and takes the value

$$\frac{\mu((t^n_{j-1}, t^n_j])}{|\mu|((t^n_{j-1}, t^n_j])} = \frac{a(t^n_j) - a(t^n_{j-1})}{|\mu|((t^n_{j-1}, t^n_j])}$$

on this interval. On the other hand, $(X_n)_{n\geq 0}$ is a closed martingale, with respect to $(\mathcal{B}_n)_{n\geq 0}$. Since $X \in \mathcal{B} = \vee_n \mathcal{B}_n$, this martingale converges to X in L^1. In particular,

$$\lim_{n\to\infty} \mathbb{E}[|X_n|] = \mathbb{E}[|X|] = 1.$$

The desired result follows by noting that

$$\mathbb{E}[|X_n|] = (|\mu|([0, T]))^{-1} \sum_{j=1}^{p_n} |a(t^n_j) - a(t^n_{j-1})|$$

and recalling $|\mu|([0, T]) = \int_0^T |da(s)|$.

Lemma 4.3

If $f : [0, T] \to \mathbb{R}$ is a continuous function, and if $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = T$ is a sequence of partitions of $[0, T]$ whose mesh goes to zero, we have

$$\int_0^T f(s)da(s) = \lim_{n\to\infty} \sum_{j=1}^{p_n} f(t_{j-1}^n)(a(t_j^n) - a(t_{j-1}^n)).$$

Proof of Lemma 4.3

Let f_n be defined on $[0, T]$ be $f_n(s) = f(t_{j-1}^n)$ if $s \in (t_{j-1}^n, t_j^n]$, $1 \le j \le p_n$, and $f_n(0) = 0$. Then

$$\sum_{j=1}^{p_n} f(t_{j-1}^n)(a(t_j^n) - a(t_{j-1}^n)) = \int_{[0,T]} f_n(s)\mu(ds)$$

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We say that a function $a : \mathbb{R}_+ \to \mathbb{R}$ is of finite variation on $\mathbb{R}_+$ if the restriction of a to $[0, T]$ is of finite variation, for every $T > 0$. Then there is a unique σ-finite (positive) measure on $\mathbb{R}_+$ whose restriction to every interval $[0, T]$ is the total variation measure of the restriction of $a|_{[0,T]}$ and we write

$$\int_0^\infty f(s)|da(s)|$$

for the integral of a non-negative Borel function f on $\mathbb{R}_+$ with respect to this σ-finite measure. Furthermore, we can define

$$\int_0^\infty f(s)da(s) = \lim_{T\to\infty} \int_0^T f(s)da(s) \in (-\infty, \infty)$$

for any real Borel function f on $\mathbb{R}_+$ with $\int_0^\infty |f(s)||da(s)| < \infty$.

We now talk about finite variation processes. We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq o}, \mathbb{P})$.

Definition 4.4

An adapted process $A = (A_t)_{t \geq 0}$ is called a finite variation process if all its sample paths are finite variation functions on $\mathbb{R}_+$. If in addition the sample paths are non-decreasing functions, the process A is called an increasing process.

In particular, $A_0 = 0$ and and the sample paths of A are continuous. We could define finite variation processes with cadlag sample paths, but in this book we consider only the case of continuous sample paths. The convention that the initial value of a finite variation process is 0 will be convenient for certain uniqueness statements.

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If A is a finite variation process, the process

$$V_t = \int_0^t |dA_s|$$

is an increasing process. Indeed, it is clear that the sample paths of V are nondecreasing functions (as well as continuous functions that vanish at $t = 0$). The fact that V_t is $\mathcal{F}_t$-measurable can be deduced from the second part of Proposition 4.2. Writing $A_t = \frac{1}{2}(V_t + A_t) - \frac{1}{2}(V_t - A_t)$ shows that any finite variation process can be written as the difference of two increasing processes (the converse is obvious).

Proposition 4.5

Let A be a finite variation process, and let H be a progressive process such that

$$\int_0^t |H_s(\omega)||dA_s(\omega)| < \infty, \quad \text{for all } t \geq 0, \omega \in \Omega.$$

Then the process $H \cdot A = ((H \cdot A)_t)_{t \geq 0}$ defined by

$$(H \cdot A)_t = \int_0^t H_s dA_s$$

is also a finite variation process.

Proof of Proposition 4.5

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Proof of Proposition 4.5 (cont)

To this end, it is enough to check that, if $t > 0$ is fixed, if $h : \Omega \times [0, t] \to \mathbb{R}$ is $\mathcal{F}_t \otimes \mathcal{B}([0, t])$-measurable, and if $\int_0^t |h(\omega, s)||dA_s(\omega)| < \infty$ for every ω, then $\int_0^t h(\omega, s)dA_s(\omega) \in \mathcal{F}_t$.

If $h(\omega, s) = 1_{(u,v]}(s)1_\Gamma(\omega)$ wit $(u, v] \subset [0, t]$ and $\Gamma \in \mathcal{F}_t$, the result is immediate since $\int_0^t h(\omega, s)dA_s(\omega) = 1_\Gamma(\omega)(A_v(\omega) - A_u(\omega))$ in this case. A monotone class argument then gives the case $h = 1_G$, $G \in \mathcal{F}_t \otimes \mathcal{B}([0, t])$. Finally, in the general case, we observe that we can write h as a pointwise limit of a sequence of simple functions h_n such that $|h_n| \leq |h|$ for every n, and that we then have $\int_0^t h_n(\omega, s)dA_s(\omega) \to \int_0^t h(\omega, s)dA_s(\omega)$ by dominated convergence, for every $\omega \in \Omega$.

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Remarks

(i) It happens frequently that instead of the assumption of the proposition we have the weaker assumption that a.s.

$$\int_0^t |H_s(\omega)||dA_s(\omega)| < \infty, \quad \text{for all } t \geq 0.$$

If the filtration is complete, we can still define $H \cdot A$ as a finite variation process under this weaker assumption.We replace H by the process H' defined by

$$H'_t(\omega) = \begin{cases} H_t(\omega), & \text{if } \int_0^n |H_s(\omega)||dA_s(\omega)| < \infty \ \ \forall n \\ 0, & \text{otherwsie.} \end{cases}$$

Thanks to the fact that the filtration is complete, the process H' is still progressive, which allows us to define $H \cdot A = H' \cdot A$. We will use this extension of Proposition 4.5 implicitly from now on.

(ii) If H and K are progressive and $\int_0^t |H_s||dA_s| < \infty$, $\int_0^t |K_s H_s||dA_s| < \infty$ for every $t \geq 0$, we hve

$$K \cdot (H \cdot A) = (KH) \cdot A.$$

This follows from the analogous deterministic result saying informally that $k(s)(h(s)\mu(ds)) = (k(s)h(s))\mu(ds)$ if h and kh are integrable with respect to the signed measure μ on $[0, t]$.

Outline

We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. If T is a stopping time, and if $X = (X_t)_{t\geq 0}$ is an adapted process with continuous sample paths, we will write X^T for process X stopped at T, defined by $X_t^T = X_{t\wedge T}$ for every $t \geq 0$. Note that if S is another stopping time,

$$(X^T)^S = (X^S)^T = X^{S\wedge T}.$$

Definition 4.6

An adapted process $M = (M_t)_{t\geq 0}$ with continuous sample paths and such that $M_0 = 0$ a.s. is called a continuous local martingale if there exists a non-decreasing sequence $(T_n)_{n\geq 0}$ of stopping times such that $T_n \uparrow \infty$ and that, for each $n \geq 0$, M^{T_n} is a uniformly integrable martingale.

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More generally, when we do not assume that $M_0 = 0$ a.s. we say that M is a continuous local martingale if the process $N_t = M_t - M_0$ is a continuous local martingale.

In all cases, we say that the sequence of stopping times $(T_n)_{n \geq 0}$ reduces M if $T_n \uparrow \infty$ and that, for each $n \geq 0$, M^{T_n} is a uniformly integrable martingale.

(i) We do not require in the definition of a continuous local martingale that $M_t \in L^1$. In particular, M_0 maybe any $\mathcal{F}_0$-random variable.

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(ii) Any martingale with continuous sample paths is a continuous local martingale, but the converse is false, and for this reason we will sometimes speak of "true martingales" to emphasize the difference with local martingales. Let us give a few examples of continuous local martingales which are not (true) martingales. If B is an $(\mathcal{F}_t)$-Brownian motion started from 0, and Z is an $\mathcal{F}_0$-random variable, the process $M_t = Z + B_t$ is always a continuous local martingale, but is not a martingale if $\mathbb{E}[|Z|] = \infty$. If we require the property $M_0 = 0$ a.s., we can also consider $M_t = ZB_t$, which is always a continuous local martingale but is not a martingale if $E[|Z|] = \infty$.