Outline
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1 3.4 Optional Stopping Theorems
### Definition 3.20
A martingale \((X_t)_{t \geq 0}\) is said to be closed if there exists a random variable \(Z \in L^1\) such that, for every \(t \geq 0\),

\[ X_t = \mathbb{E}[Z | \mathcal{F}_t]. \]

### Theorem 3.21
Let \(X\) be a martingale with right-continuous sample paths. Then the following properties are equivalent:

(i) \(X\) is closed;

(ii) \((X_t)_{t \geq 0}\) is uniformly integrable;

(iii) \(X_t\) converges a.s. and in \(L^1\) as \(t \to \infty\).

Moreover, if these properties hold, we have \(X_t = \mathbb{E}[X_\infty | \mathcal{F}_t]\) for every \(t \geq 0\), where \(X_\infty \in L^1\) is the a.s. limit of \(X_t\) as \(t \to \infty\).
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Proof of Theorem 3.21

(i)⇒(ii) is trivial.

(ii)⇒(iii) If (ii) holds, then $(X_t)_{t \geq 0}$ is bounded in $L^1$ and Proposition 3.19 implies that $X_t$ converges a.s.. By uniform integrability, the convergence also holds in $L^1$.

(iii)⇒(i) If (iii) holds, for every $s \geq 0$, we can pass to the limit $t \to \infty$ in the equality $X_s = \mathbb{E}[X_t|\mathcal{F}_s]$ (using the fact that the conditional expectation is continuous for the $L^1$-norm), and we get $X_s = \mathbb{E}[X_\infty|\mathcal{F}_s]$. 
We will now use the optional stopping theorems for discrete martingales and supermartingales in order to establish similar results in the continuous time setting. Let $(X_t)_{t \geq 0}$ be a martingale or a supermartingale with right-continuous sample paths, and such that $X_t$ converges a.s. as $t \uparrow \infty$ to a random variable denoted by $X_\infty$. Then, for every stopping time $T$, we write $X_T$ for the random variable

$$X_T(\omega) = 1\{T(\omega) < \infty\} X_T(\omega) + 1\{T(\omega) = \infty\} X_\infty(\omega).$$

$X_T(\omega)$ is defined for every $\omega \in \Omega$ and is $\mathcal{F}_T$-measurable.
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\(X_T(\omega)\) is defined for every \(\omega \in \Omega\) and is \(\mathcal{F}_T\)-measurable.
Theorem 3.22 (optional Stopping Theorem)

Let \((X_t)_{t \geq 0}\) be a uniformly integrable martingale with right-continuous sample paths. Let \(S\) and \(T\) be two stopping times with \(S \leq T\). Then \(X_S\) and \(X_T\) are in \(L^1\) and

\[ X_S = \mathbb{E}[X_T | \mathcal{F}_S]. \]

In particular, for every stopping time \(S\), we have

\[ X_S = \mathbb{E}[X_\infty | \mathcal{F}_S] \]

and

\[ \mathbb{E}[X_S] = \mathbb{E}[X_\infty] = \mathbb{E}[X_0]. \]
Proof of Theorem 3.22

For every integer \( n \geq 0 \), define

\[
T_n = \sum_{k=0}^{\infty} \frac{k + 1}{2^n} 1\{k2^{-n} < T \leq (k+1)2^{-n}\} + \infty \cdot 1\{T=\infty\}
\]

and

\[
S_n = \sum_{k=0}^{\infty} \frac{k + 1}{2^n} 1\{k2^{-n} < S \leq (k+1)2^{-n}\} + \infty \cdot 1\{S=\infty\}
\]

Then \((T_n)\) and \((S_n)\) are two sequences of stopping times that decrease respectively to \(T\) and to \(S\). Moreover, we have \(S_n \leq T_n\) for every \( n \geq 0 \).

Note that, for every fixed \( n \), \(2^n S_n\) and \(2^n T_n\) are stopping times of the discrete filtration \(\mathcal{H}_k^{(n)} := \mathcal{F}_{k2^{-n}}\), and \(Y_k^{(n)} := X_{k2^{-n}}\) is a discrete martingale with respect to this filtration.
Proof of Theorem 3.22 (cont)

From the optional stopping theorem for uniformly integrable discrete martingales we get that $Y_{2^n S_n}^{(n)}$ and $Y_{2^n T_n}^{(n)}$ are in $L^1$, and

$$X_{S_n} = Y_{2^n S_n}^{(n)} = \mathbb{E}[Y_{2^n T_n}^{(n)} | H_{2^n S_n}^{(n)}] = \mathbb{E}[X_{T_n} | F_{S_n}].$$

(note that $H_{2^n S_n}^{(n)} = F_{S_n}$.)

Let $A \in F_S$. Since $F_S \subset F_{S_n}$, we have $A \in F_{S_n}$ and thus

$$\mathbb{E}[1_A X_{S_n}] = \mathbb{E}[1_A X_{T_n}].$$

By the right-continuity of sample paths, we get a.s.

$$X_S = \lim_{n \to \infty} X_{S_n}, \quad X_T = \lim_{n \to \infty} X_{T_n}.$$

These limits also hold in $L^1$. Indeed, thanks again to the optional stopping theorem for uniformly integrable discrete martingales, we have $X_{S_n} = \mathbb{E}[X_\infty | F_{S_n}]$ for every $n$, and thus $(X_{S_n})$ is uniformly integrable (the same is true for $(X_{T_n})$).
Proof of Theorem 3.22 (cont)

The $L^1$-convergence implies that $X_S$ and $X_T$ are in $L^1$, and also allows us to pass to the limit $n \to \infty$ in the equality $\mathbb{E}[1_A X_{S_n}] = \mathbb{E}[1_A X_{T_n}]$ to get

$$\mathbb{E}[1_A X_S] = \mathbb{E}[1_A X_T].$$

Since this holds for every $A \in \mathcal{F}_S$, and since $X_S$ is $\mathcal{F}_S$-measurable, we conclude that

$$X_S = \mathbb{E}[X_T|\mathcal{F}_S]$$

which completes the proof.
Corollary 3.23

Let \((X_t)_{t \geq 0}\) be a martingale with right-continuous sample paths, and let \(S \leq T\) be two bounded stopping times. Then \(X_S\) and \(X_T\) are in \(L^1\), and

\[X_S = \mathbb{E}[X_T | \mathcal{F}_S].\]

Proof of Corollary 3.23

Let \(a \geq 0\) be such that \(S \leq T \leq a\). We apply Theorem 3.22 to the martingale \((X_{t \wedge a})_{t \geq 0}\) which is closed by \(X_a\).
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Corollary 3.24

Let \((X_t)_{t \geq 0}\) be a martingale with right-continuous sample paths, and let \(T\) be a stopping time.

(i) The process \((X_t \land T)_{t \geq 0}\) is still a martingale.

(ii) Suppose in addition that the martingale \((X_t)_{t \geq 0}\) is uniformly integrable. Then the process \((X_t \land T)_{t \geq 0}\) is also a uniformly integrable martingale, and more precisely we have for every \(t \geq 0\),

\[ X_{t \land T} = \mathbb{E}[X_T | \mathcal{F}_t]. \] (1)

Proof of Corollary 3.24

It suffices to prove (ii). Once (ii) is proven, we just need to apply (ii) to the (uniformly integrable) martingale \((X_t \land a)_{t \geq 0}\) for every \(a \geq 0\).
Corollary 3.24
Let \((X_t)_{t \geq 0}\) be a martingale with right-continuous sample paths, and let \(T\) be a stopping time.

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Proof of Corollary 3.24 (cont)

(ii) Note that \( t \land T \) is a stopping time. By Theorem 3.22, \( X_{t \land T} \) and \( X_T \) are in \( L^1 \), \( X_{t \land T} \) is \( \mathcal{F}_{t \land T} \)-measurable, and hence \( \mathcal{F}_t \)-measurable. In order to get (1), it suffices to show that, for every \( A \in \mathcal{F}_t \),

\[
\mathbb{E}[1_A X_T] = \mathbb{E}[1_A X_{t \land T}].
\]

Let us fix \( A \in \mathcal{F}_t \). First, we have trivially

\[
\mathbb{E}[1_A \cap \{T \leq t\} X_T] = \mathbb{E}[1_A \cap \{T \leq t\} X_{t \land T}]. \tag{2}
\]

On the other hand, by Theorem 3.22, we have

\[
X_{t \land T} = \mathbb{E}[X_T | \mathcal{F}_{t \land T}].
\]

and we notice that we have both \( A \cap \{T > t\} \in \mathcal{F}_t \) and \( A \cap \{T > t\} \in \mathcal{F}_T \), and so \( A \cap \{T > t\} \in \mathcal{F}_{t \land T} \). Thus,

\[
\mathbb{E}[1_A \cap \{T > t\} X_T] = \mathbb{E}[1_A \cap \{T > t\} X_{t \land T}].
\]

By adding this equality to (2), we get the desired result.
Applications

The optional stopping theorem is useful for explicit calculations of probability distributions. Now I give a few important and typical examples of such applications. Let $B$ be a real-valued Brownian motion started from 0. $B$ is a martingale with continuous sample paths with respect to its canonical filtration. For every $a \in \mathbb{R}$, set $T_a = \inf\{ t \geq 0 : B_t = a \}$. Recall that $T_a < \infty$ a.s.

(a) For every $a < 0 < b$, we have

$$P(T_a < T_b) = \frac{b}{b - a}, \quad P(T_b < T_a) = \frac{-a}{b - a}.$$
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P(T_a < T_b) = \frac{b}{b-a}, \quad P(T_b < T_a) = \frac{-a}{b-a}.
$$
Let $T = T_a \wedge T_b$ and consider the stopped martingale $M_t = B_{t \wedge T}$. Then $|M_t|$ is bounded above by $b \vee |a|$, and the martingale $M$ is thus uniformly integrable. We can apply Theorem 3.22 and we get

$$0 = \mathbb{E}[M_0] = \mathbb{E}[M_T] = b \mathbb{P}(T_b < T_a) + a \mathbb{P}(T_a < T_b).$$

Since we also have $\mathbb{P}(T_b < T_a) + \mathbb{P}(T_a < T_b) = 1$, the result follows.

The result remains valid if we replace Brownian motion by a martingale with continuous sample paths and initial value 0, provided we know that this process exits $(a, b)$ a.s.
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The result remains valid if we replace Brownian motion by a martingale with continuous sample paths and initial value 0, provided we know that this process exits $(a, b)$ a.s.
(b) For every $a > 0$, consider the stopping time $U_a = \inf\{t \geq 0 : |B_t| = a\}$. Then

$$E[U_a] = a^2.$$

Consider the martingale $M_t = B_t^2 - t$. By Corollary 3.24, $M_{t \wedge U_a}$ is still a martingale, and therefore $E[M_{t \wedge U_a}] = E[M_0] = 0$, giving $E[B_{t \wedge U_a}^2] = E[t \wedge U_a]$. By the monotone convergence theorem, we have $E[t \wedge U_a] \uparrow E[U_a]$. By the dominated convergence theorem, we have $E[B_{t \wedge U_a}^2] = E[B_{U_a}^2] = a^2$. The result follows.
(b) For every $a > 0$, consider the stopping time
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Consider the martingale $M_t = B^2_t - t$. By Corollary 3.24, $M_{t\wedge U_a}$ is still a martingale, and therefore $\mathbb{E}[M_{t\wedge U_a}] = \mathbb{E}[M_0] = 0$, giving

$$\mathbb{E}[B^2_{t\wedge U_a}] = \mathbb{E}[t \wedge U_a].$$

By the monotone convergence theorem, we have $\mathbb{E}[t \wedge U_a] \uparrow \mathbb{E}[U_a]$. By the dominated convergence theorem, we have $\mathbb{E}[B^2_{t\wedge U_a}] = \mathbb{E}[B^2_{U_a}] = a^2$. The result follows.
(c) For any $a > 0$, it holds that

$$\mathbb{E}[e^{-\lambda T_a}] = e^{-a\sqrt{2\lambda}}, \quad \lambda > 0.$$ 

For any $\lambda > 0$, consider the exponential martingale

$$N_t^\lambda = \exp\left(\lambda B_t - \frac{\lambda^2}{2} t\right).$$

By Corollary 3.24, the stopped process $N_{t \wedge T_a}^\lambda$ is a martingale, and we immediately see that this martingale is bounded above by $e^{\lambda a}$, hence uniformly integrable. By applying the last assertion of Theorem 3.22 to this martingale and to the stopping time $S = T_a$ (or to $S = \infty$), we get

$$e^{\lambda a} \mathbb{E}[e^{-\frac{\lambda^2}{2} T_a}] = \mathbb{E}[N_{T_a}^\lambda] = \mathbb{E}[N_0^\lambda] = 1.$$ 

Replacing $\lambda$ by $\sqrt{2\lambda}$, we immediately get the desired assertion.
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