Math 562 Fall 2020

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Outline

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Definition 3.20

A martingale $(X_t)_{t\geq 0}$ is said to be closed if there exists a random variable $Z \in L^1$ such that, for every $t \geq 0$,

$$X_t = \mathbb{E}[Z|\mathcal{F}_t].$$

Theorem 3.21

Let X be a martingale with right-continuous sample paths. Then the following properties are equivalent:

(i) X is closed;

(ii) $(X_t)_{t\geq 0}$ is uniformly integrable;

(iii) X_t converges a.s. and in L^1 as $t \to \infty$.

Moreover, if these properties hold, we have $X_t = \mathbb{E}[X_\infty|\mathcal{F}_t]$ for every $t \geq 0$, where $X_\infty \in L^1$ is the a.s. limit of X_t as $t \to \infty$.

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Proof of Theorem 3.21

(i)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(iii) If (ii) holds, then $(X_t)_{t\geq 0}$ is bounded in L^1 and Proposition 3.19 implies that X_t converges a.s.. By uniform integrability, the convergence also holds in L^1.

(iii)$\Rightarrow$(i) If (iii) holds, for every $s \geq 0$, we can pass to the limit $t \to \infty$ in the equality $X_s = \mathbb{E}[X_t|\mathcal{F}_s]$ (using the fact that the conditional expectation is continuous for the L^1-norm), and we get $X_s = \mathbb{E}[X_\infty|\mathcal{F}_s]$.

We will now use the optional stopping theorems for discrete martingales and supermartingales in order to establish similar results in the continuous time setting. Let $(X_t)_{t\geq 0}$ be a martingale or a supermartingale with right-continuous sample paths, and such that X_t converges a.s. as $t \uparrow \infty$ to a random variable denoted by X_∞. Then, for every stopping time T, we write X_T for the random variable

$$X_T(\omega) = 1_{\{T(\omega)<\infty\}} X_{T(\omega)}(\omega) + 1_{\{T(\omega)=\infty\}} X_\infty(\omega).$$

$X_T(\omega)$ is defined for every $\omega \in \Omega$ and is $\mathcal{F}_T$-measurable.

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Theorem 3.22 (optional Stopping Theorem)

Let $(X_t)_{t\geq 0}$ be a uniformly integrable martingale with right-continuous sample paths. Let S and T be two stopping times with $S \leq T$. Then X_S and X_T are in L^1 and

$$X_S = \mathbb{E}[X_T | \mathcal{F}_S].$$

In particular, for every stopping time S, we have

$$X_S = \mathbb{E}[X_\infty | \mathcal{F}_S]$$

and

$$\mathbb{E}[X_S] = \mathbb{E}[X_\infty] = \mathbb{E}[X_0].$$

Proof of Theorem 3.22

For every integer $n \geq 0$, define

$$T_n = \sum_{k=0}^{\infty} \frac{k+1}{2^n} 1_{\{k2^{-n} < T \leq (k+1)2^{-n}\}} + \infty \cdot 1_{\{T=\infty\}}$$

and

$$S_n = \sum_{k=0}^{\infty} \frac{k+1}{2^n} 1_{\{k2^{-n} < S \leq (k+1)2^{-n}\}} + \infty \cdot 1_{\{S=\infty\}}$$

Then (T_n) and (S_n) are two sequences of stopping times that decrease respectively to T and to S. Moreover, we have $S_n \leq T_n$ for every $n \geq 0$.

Note that, for every fixed n, $2^n S_n$ and $2^n T_n$ are stopping times of the discrete filtration $\mathcal{H}_k^{(n)} := \mathcal{F}_{k2^{-n}}$, and $Y_k^{(n)} := X_{k2^{-n}}$ is a discrete martingale with respect to this filtration.

Proof of Theorem 3.22 (cont)

From the optional stopping theorem for uniformly integrable discrete martingales we get that $Y^{(n)}_{2^n S_n}$ and $Y^{(n)}_{2^n T_n}$ are in L^1, and

$$X_{S_n} = Y^{(n)}_{2^n S_n} = \mathbb{E}[Y^{(n)}_{2^n T_n} | H^{(n)}_{2^n S_n}] = \mathbb{E}[X_{T_n} | \mathcal{F}_{S_n}].$$

(note that $\mathcal{H}^{(n)}_{2^n S_n} = \mathcal{F}_{S_n}$.)
Let $A \in \mathcal{F}_S$. Since $\mathcal{F}_S \subset \mathcal{F}_{S_n}$, we have $A \in \mathcal{F}_{S_n}$ and thus

$$\mathbb{E}[1_A X_{S_n}] = \mathbb{E}[1_A X_{T_n}].$$

By the right-continuity of sample paths, we get a.s.

$$X_S = \lim_{n\to\infty} X_{S_n}, \quad X_T = \lim_{n\to\infty} X_{T_n}.$$

These limits also hold in L^1. Indeed, thanks again to the optional stopping theorem for uniformly integrable discrete martingales, we have $X_{S_n} = \mathbb{E}[X_\infty | \mathcal{F}_{S_n}]$ for every n, and thus (X_{S_n}) is uniformly integrable (the same is true for (X_{T_n})).

Proof of Theorem 3.22 (cont)

The L^1-convergence implies that X_S and X_T are in L^1, and also allows us to pass to the limit $n \to \infty$ in the equality $\mathbb{E}[1_A X_{S_n}] = \mathbb{E}[1_A X_{T_n}]$ to get

$$\mathbb{E}[1_A X_S] = \mathbb{E}[1_A X_T].$$

Since this holds for every $A \in \mathcal{F}_S$, and since X_S is $\mathcal{F}_S$-measurable, we conclude that

$$X_S = \mathbb{E}[X_T | \mathcal{F}_S]$$

which completes the proof.

Corollary 3.23

Let $(X_t)_{t\geq 0}$ be a martingale with right-continuous sample paths, and let $S \leq T$ be two bounded stopping times. Then X_S and X_T are in L^1, and

$$X_S = \mathbb{E}[X_T|\mathcal{F}_S].$$

Proof of Corollary 3.23

Let $a \geq 0$ be such that $S \leq T \leq a$. We apply Theorem 3.22 to the martingale $(X_{t\wedge a})_{t\geq 0}$ which is closed by X_a.

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Corollary 3.24

Let $(X_t)_{t\geq 0}$ be a martingale with right-continuous sample paths, and let T be a stopping time.

(i) The process $(X_{t\wedge T})_{t\geq 0}$ is still a martingale.

(ii) Suppose in addition that the martingale $(X_t)_{t\geq 0}$ is uniformly integrable. Then the process $(X_{t\wedge T})_{t\geq 0}$ is also a uniformly integrable martingale, and more precisely we have for every $t \geq 0$,

$$X_{t\wedge T} = \mathbb{E}[X_T|\mathcal{F}_t]. \tag{1}$$

Proof of Corollary 3.24

It suffices to prove (ii). Once (ii) is proven, we just need to apply (ii) to the (uniformly integrable) martingale $(X_{t\wedge a})_{t\geq 0}$ for every $a \geq 0$.

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Proof of Corollary 3.24 (cont)

(ii) Note that $t \wedge T$ is a stopping time. By Theorem 3.22, $X_{t\wedge T}$ and X_T are in L^1, $X_{t\wedge T}$ is $\mathcal{F}_{t\wedge T}$-measurable, and hence $\mathcal{F}_t$-measurable. In order to get (1), it suffices to show that, for every $A \in \mathcal{F}_t$,

$$\mathbb{E}[1_A X_T] = \mathbb{E}[1_A X_{t\wedge T}].$$

Let us fix $A \in \mathcal{F}_t$. First, we have trivially

$$\mathbb{E}[1_{A\cap\{T\leq t\}} X_T] = \mathbb{E}[1_{A\cap\{T\leq t\}} X_{t\wedge T}]. \quad (2)$$

On the other hand, by Theorem 3.22, we have

$$X_{t\wedge T} = \mathbb{E}[X_T|\mathcal{F}_{t\wedge T}].$$

and we notice that we have both $A \cap \{T > t\} \in \mathcal{F}_t$ and $A \cap \{T > t\} \in \mathcal{F}_T$, and so $A \cap \{T > t\} \in \mathcal{F}_{t\wedge T}$. Thus,

$$\mathbb{E}[1_{A\cap\{T>t\}} X_T] = \mathbb{E}[1_{A\cap\{T>t\}} X_{t\wedge T}].$$

By adding this equality to (2), we get the desired result.

Applications

The optional stopping theorem is useful for explicit calculations of probability distributions. Now I give a few important and typical examples of such applications. Let B be a real-valued Brownian motion started from 0. B is a martingale with continuous sample paths with respect to its canonical filtration. For every $a \in \mathbb{R}$, set $T_a = \inf\{t \geq 0 : B_t = a\}$. Recall that $T_a < \infty$ a.s.

(a) For every $a < 0 < b$, we have

$$\mathbb{P}(T_a < T_b) = \frac{b}{b-a}, \quad \mathbb{P}(T_b < T_a) = \frac{-a}{b-a}.$$

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$$\mathbb{P}(T_a < T_b) = \frac{b}{b-a}, \quad \mathbb{P}(T_b < T_a) = \frac{-a}{b-a}.$$

Let $T = T_a \wedge T_b$ and consider the stopped martingale $M_t = B_{t \wedge T}$. Then $|M_t|$ is bounded above by $b \vee |a|$, and the martingale M is thus uniformly integrable. We can apply Theorem 3.22 and we get

$$0 = \mathbb{E}[M_0] = \mathbb{E}[M_T] = b\mathbb{P}(T_b < T_a) + a\mathbb{P}(T_a < T_b).$$

Since we also have $\mathbb{P}(T_b < T_a) + \mathbb{P}(T_a < T_b) = 1$, the result follows.

The result remains valid if we replace Brownian motion by a martingale with continuous sample paths and initial value 0, provided we know that this process exits (a, b) a.s.

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The result remains valid if we replace Brownian motion by a martingale with continuous sample paths and initial value 0, provided we know that this process exits (a, b) a.s.

(b) For every $a > 0$, consider the stopping time $U_a = \inf\{t \geq 0 : |B_t| = a\}$. Then

$$\mathbb{E}[U_a] = a^2.$$

Consider the martingale $M_t = B_t^2 - t$. By Corollary 3.24, $M_{t \wedge U_a}$ is still a martingale, and therefore $\mathbb{E}[M_{t \wedge U_a}] = \mathbb{E}[M_0] = 0$, giving $\mathbb{E}[B^2_{t \wedge U_a}] = \mathbb{E}[t \wedge U_a]$. By the monotone convergence theorem, we have $\mathbb{E}[t \wedge U_a] \uparrow \mathbb{E}[U_a]$. By the dominated convergence theorem, we have $\mathbb{E}[B^2_{t \wedge U_a}] = \mathbb{E}[B^2_{U_a}] = a^2$. The result follows.

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(c) For any $a > 0$, it holds that

$$\mathbb{E}[e^{-\lambda T_a}] = e^{-a\sqrt{2\lambda}}, \quad \lambda > 0.$$

For any $\lambda > 0$, consider the exponential martingale

$$N_t^\lambda = \exp\left(\lambda B_t - \frac{\lambda^2}{2} t\right).$$

By Corollary 3.24, the stopped process $N_{t \wedge T_a}^\lambda$ is a martingale, and we immediately see that this martingale is bounded above by $e^{\lambda a}$, hence uniformly integrable. By applying the last assertion of Theorem 3.22 to this martingale and to the stopping time $S = T_a$ (or to $S = \infty$), we get

$$e^{\lambda a}\mathbb{E}[e^{-\frac{\lambda^2}{2} T_a}] = \mathbb{E}[N_{T_a}^\lambda] = \mathbb{E}[N_0^\lambda] = 1.$$

Replacing λ by $\sqrt{2\lambda}$, we immediately get the desired assertion.

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