Math 562 Fall 2020

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Outline

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Lemma 3.16

Let D be a countable dense subset of $\mathbb{R}_+$ and let f be a real function defined on D. We assume that, for every $T \in D$,

(i) f is bounded on $D \cap [0, T]$,

(ii) for all rationals a and b such that $a < b$,

$$M_{ab}^f(D \cap [0, T]) < \infty.$$

Then, the right-limit

$$f(t+) := \lim_{s \downarrow\downarrow t, s \in D} f(s)$$

exists for every real $t \geq 0$, and similarly the left-limit

$$f(t-) := \lim_{s \uparrow\uparrow t, s \in D} f(s)$$

exists for every real $t > 0$. Furthermore, the function $g : \mathbb{R}_+ \to \mathbb{R}$ defined by $g(t) = f(t+)$ is cadlag.

Theorem 3.17

Let $(X_t)_{t\geq 0}$ be a supermartingale, and let D be a countable dense subset of $\mathbb{R}_+$.

(i) For almost every $\omega \in \Omega$, the restriction of the function $s \to X_s(\omega)$ to D has a right-limit

$$X_{t+}(\omega) := \lim_{s\downarrow\downarrow t, s\in D} X_s(\omega), \quad \forall t \in [0, \infty),$$

and a left-limit

$$X_{t-}(\omega); = \lim_{s\uparrow\uparrow t, s\in D} X_s(\omega), \quad \forall t \in (0, \infty).$$

(ii) For every $t \geq 0$, $X_{t+} \in L^1$ and

$$X_t \geq \mathbb{E}[X_{t+}|\mathcal{F}_t]$$

with equality if the function $t \mapsto \mathbb{E}[X_t]$ is right-continuous. The process $(X_{t+})_{t\geq 0}$ is a supermartingale with respect to $(\mathcal{F}_{t+})$. It is a martingale if X is a martingale.

Proof of Theorem 3.17

(i) Fix $T \in D$. By the remark following Proposition 3.15, we have

$$\sup_{s \in D \cap [0,T]} |X_s| < \infty, \quad a.s$$

As in the proof of Proposition 3.15, we can choose a sequence $(D_m)_{m \geq 1}$ of finite subsets of D that increase to $D \cap [0, T]$ and are such that $0, T \in D_m$. Doob's upcrossing inequality for discrete supermartingales gives, for every $a < b$ and every $m \geq 1$,

$$\mathbb{E}[M_{ab}^X(D_m)] \leq \frac{1}{b-a}\mathbb{E}[(X_T - a)^-].$$

We let $m \uparrow \infty$ and get by monotone convergence

$$\mathbb{E}[M_{ab}^X(D \cap [0, T])] \leq \frac{1}{b-a}\mathbb{E}[(X_T - a)^-] < \infty.$$

Proof of Theorem 3.17 (cont)

We thus have

$$M_{ab}^{X}(D \cap [0, T]) < \infty, \quad a.s.$$

Define N to be the event

$$\cup_{T \in D} \left(\{ \sup_{t \in D[0,T]} |X_t| = \infty \} \cup \left(\cup_{a,b \in \mathbb{Q}, a<b} \{ M_{ab}^{X}(D \cap [0, T]) = \infty \} \right) \right) \tag{1}$$

Then $\mathbb{P}(N) = 0$ by the preceding considerations. On the other hand, if $\omega \notin N$, the function $D \ni t \mapsto X_t(\omega)$ satisfies all assumptions of Lemma 3.16. Assertion (i) now follows from this lemma.

Proof of Theorem 3.17 (cont)

(ii) For every $\omega \in \Omega$, not just for $\omega \notin N$, we define

$$X_{t+}(\omega) = \begin{cases} \lim_{s\downarrow\downarrow t, s\in D} X_s(\omega), & \text{if the limit exists} \\ 0, & \text{otherwise.} \end{cases}$$

With this definition, X_{t+} is $\mathcal{F}_{t+}$-measurable.

Fix $t \geq 0$ and choose a sequence $(t_n)_{n\geq 0}$ in D such that t_n decreases strictly to t as $n \uparrow \infty$. Then, by construction, we have a.s.

$$X_{t+} = \lim_{n\uparrow\infty} X_{t_n}.$$

Set $Y_k = X_{t_{-k}}$ for $k \leq 0$. Then Y is a backward supermartingale with respect to the filtration $\mathcal{H}_k = \mathcal{F}_{t_{-k}}$. By Proposition 3.13, we have $\sup_{k\leq 0} \mathbb{E}[|Y_k|] < \infty$. Convergence theorem for backward supermartingales implies that X_{t_n} converges to X_{t+} in L^1. Thus $X_{t+} \in L^1$.

Proof of Theorem 3.17 (cont)

Thanks to the L^1-convergence, we can let $n \uparrow \infty$ in the inequality $X_t \geq \mathbb{E}[X_{t_n}|\mathcal{F}_t]$ to get

$$X_t \geq \mathbb{E}[X_{t+}|\mathcal{F}_t].$$

Use the L^1 convergence again, we get $\mathbb{E}[X_{t+}] = \lim_{n\to\infty} \mathbb{E}[X_{t_n}]$. Thus, if the function $s \mapsto \mathbb{E}[X_s]$ is right-continuous, we must have $E[X_t] = \mathbb{E}[X_{t+}] = \mathbb{E}[\mathbb{E}[X_{t+}|\mathcal{F}_t]]$, and the inequality $X_t \geq \mathbb{E}[X_{t+}|\mathcal{F}_t]$ then forces $X_t = \mathbb{E}[X_{t+}|\mathcal{F}_t]$.

Proof of Theorem 3.17 (cont)

We have already noticed that X_{t+} is $\mathcal{F}_{t+}$-measurable. Let $s < t$ and let $(s_n)_{n\geq 0}$ be a sequence in D that decreases strictly to s. We may assume that $s_n \leq t_n$ for every n. Then as previously X_{s_n} converges to X_{s+} in L^1, and thus, if $A \in \mathcal{F}_{s+}$ (which implies $A \in \mathcal{F}_{s_n}$ for every n), we have

$$\mathbb{E}[X_{s+}1_A] = \lim_{n\uparrow\infty} \mathbb{E}[X_{s_n}1_A] \geq \lim_{n\uparrow\infty} \mathbb{E}[X_{t_n}1_A] = \mathbb{E}[X_{t+}1_A] = \mathbb{E}[\mathbb{E}[X_{t+}|\mathcal{F}_{s+}]1_A].$$

Since this inequality holds for every $A \in \mathcal{F}_{s+}$, and since X_{s+} and $\mathbb{E}[X_{t+}|\mathcal{F}_{s+}]$ are both $\mathcal{F}_{s+}$-measurable, it follows that $X_{s+} \geq \mathbb{E}[X_{t+}|\mathcal{F}_{s+}]$. Finally, if X is a martingale, inequalities can be replaced by equalities in the previous considerations.

Theorem 3.18

Assume that the filtration $(\mathcal{F}_t)$ is right-continuous and complete. Let $X = (X_t)_{t \geq 0}$ be a supermartingale such that the function $t \mapsto \mathbb{E}[X_t]$ is right continuous. Then X has a modification with cadlag sample paths, which is also an $(\mathcal{F}_t)$-supermartingale.

Proof of Theorem 3.18

Let D be a countable dense subset of $\mathbb{R}_+$ as in Theorem 3.17. Let N be the negligible set defined in (1). We set, for every $t \geq 0$,

$$Y_t(\omega) = \begin{cases} X_{t+}(\omega) & \text{if } \omega \notin N, \\ 0, & \text{if } \omega \in N. \end{cases}$$

Lemma 3.16 then shows that the sample paths of Y are cadlag.

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Proof of Theorem 3.18 (cont)

The random variable X_{t+} is $\mathcal{F}_{t+}$-measurable, and thus $\mathcal{F}_t$-measurable since the filtration is right-continuous. As the negligible set N belongs to $\mathcal{F}_\infty$, the completeness of the filtration ensures that Y_t is $\mathcal{F}_t$-measurable. By Theorem 3.17 (ii), we have for every $t \geq 0$,

$$X_t = \mathbb{E}[X_{t+}|\mathcal{F}_t] = X_{t+} = Y_t, \quad a.s.$$

because X_{t+} is $\mathcal{F}_t$-measurable. Consequently, Y is a modification of X. The process Y is adapted to the filtration $(\mathcal{F}_t)$. Since Y is a modification of X the inequality $\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s$, for $0 \leq s < t$, implies that the same inequality holds for Y.

Remarks

(i) The assumption that the filtration is right-continuous is necessary. Take $\Omega = \{-1, 1\}$ with the probability $\mathbb{P}$ defined by $\mathbb{P}(\{-1\}) = \mathbb{P}(\{1\}) = \frac{1}{2}$. Let ϵ be the random variable $\epsilon(\omega) = \omega$, and let the process $(X_t)_{t\geq 0}$ be defined by $X_t = 0$ if $t \in [0, 1]$, and $X_t = \epsilon$ if $t > 1$. $(X_t)_{t\geq 0}$ is a martingale wrt its canonical filtration $(\mathcal{F}_t^X)$ (which is complete since there are no nonempty negligible sets!). On the other hand, no modification of X can be right-continuous at $t = 1$. This does not contradict the theorem since the filtration is not right-continuous ($\mathcal{F}_{1+}^X \neq \mathcal{F}_1^X$).

(ii) The right-continuity of $t \mapsto \mathbb{E}[X_t]$ is also necessary. Take $X_t = f(t)$, where f is any non-increasing deterministic function. If f is not right-continuous, no modification of X can have right-continuous sample paths.

Outline

Theorem 3.19

Let X be a supermartingale with right-continuous sample paths. Assume that $(X_t)_{t\geq 0}$ is bounded in L^1. Then there exists a random variable $X_\infty \in L^1$ such that

$$\lim_{t\to\infty} X_t = X_\infty, \quad a.s.$$

Proof of Theorem 3.19

Let D be a countable dense subset of $\mathbb{R}_+$. From the proof of Theorem 3.17, we have, for every $T \in D$ and $a < b$,

$$\mathbb{E}[M_{ab}^X(D \cap [0,T])] \leq \frac{1}{b-a}\mathbb{E}[(X_T - a)^-].$$

By monotone convergence, we get, for every $a < b$,

$$\mathbb{E}[M_{ab}^X(D)] \leq \frac{1}{b-a}\sup_{t\geq 0}\mathbb{E}[(X_t - a)^-] < \infty$$

since $(X_t)_{t\geq 0}$ is bounded in L^1. Hence, a.s. for all rationals $a < b$,

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Proof of Theorem 3.19

Let D be a countable dense subset of $\mathbb{R}_+$. From the proof of Theorem 3.17, we have, for every $T \in D$ and $a < b$,

$$\mathbb{E}[M^X_{ab}(D \cap [0, T])] \leq \frac{1}{b-a}\mathbb{E}[(X_T - a)^-].$$

By monotone convergence, we get, for every $a < b$,

$$\mathbb{E}[M^X_{ab}(D)] \leq \frac{1}{b-a}\sup_{t\geq 0}\mathbb{E}[(X_t - a)^-] < \infty$$

since $(X_t)_{t\geq 0}$ is bounded in L^1. Hence, a.s. for all rationals $a < b$,

Proof of Theorem 3.19 (cont)

we have $M_{ab}^X(D) < \infty$. This implies that the limit

$$X_\infty := \lim_{D \ni t \to \infty} X_t \tag{2}$$

exists a.s. in $[-\infty, \infty]$. We can in fact exclude the values ∞ and $-\infty$, since Fatou's lemma gives

$$\mathbb{E}[|X_\infty|] \leq \liminf_{D \ni t \to \infty} \mathbb{E}[|X_t|] < \infty,$$

and we get that $X_\infty \in L^1$. The right-continuity of sample paths allows us to remove the restriction $t \in D$ in the limit (2).

Under the assumptions of Theorem 3.19, the convergence of X_t to X_∞ may not hold in L^1. The next result gives, in the case of a martingale, necessary and sufficient conditions for the convergence to also hold in L^1.

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Under the assumptions of Theorem 3.19, the convergence of X_t to X_∞ may not hold in L^1. The next result gives, in the case of a martingale, necessary and sufficient conditions for the convergence to also hold in L^1.

Definition 3.20

A martingale $(X_t)_{t\geq 0}$ is said to be closed if there exists a random variable $Z \in L^1$ such that, for every $t \geq 0$,

$$X_t = \mathbb{E}[Z|\mathcal{F}_t].$$

Theorem 3.21

Let X be a martingale with right-continuous sample paths. Then the following properties are equivalent:

(i) X is closed;

(ii) $(X_t)_{t\geq 0}$ is uniformly integrable;

(iii) X_t converges a.s. and in L^1 as $t \to \infty$.

Moreover, if these properties hold, we have $X_t = \mathbb{E}[X_\infty|\mathcal{F}_t]$ for every $t \geq 0$, where $X_\infty \in L^1$ is the a.s. limit of X_t as $t \to \infty$.

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Proof of Theorem 3.21

(i)⇒(ii) is trivial.

(ii)⇒(iii) If (ii) holds, then $(X_t)_{t\geq 0}$ is bounded in L^1 and Proposition 3.19 implies that X_t converges a.s. in L^1. By uniform integrability, the convergence also holds in L^1.

(iii)⇒(i) If (iii) holds, for every $s \geq 0$, we can pass to the limit $t \to \infty$ in the equality $X_s = \mathbb{E}[X_t|\mathcal{F}_s]$ (using the fact that the conditional expectation is continuous for the L^1-norm), and we get $X_s = \mathbb{E}[X_\infty|\mathcal{F}_s]$.