Math 562 Fall 2020

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September 18, 2020

Outline

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HW2 is due today at noon. Make sure that your HW is uploaded successfully.

Slides and videos of the lectures are also available from the Moodle page.

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Proposition 3.12

Let $(X_t)_{t\geq 0}$ be an adapted process and let $f : \mathbb{R} \to \mathbb{R}$ be a convex function such that $\mathbb{E}[|f(X_t)|] < \infty$ for all $t \geq 0$.

(i) If $(X_t)_{t\geq 0}$ is a martingale, then $(f(X_t))$ is a submartingale;

(ii) If $(X_t)_{t\geq 0}$ is a submartingale, and if in addition f is non-decreasing, then $(f(X_t))$ is a submartingale.

Proof of Proposition 3.12

By Jensen's inequality, for any $0 \leq s \leq t$,

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] \geq f(\mathbb{E}[X_t|\mathcal{F}_s]) \geq f(X_s).$$

In the last inequality, we need the fact that f is nondecreasing when $(X_t)_{t\geq 0}$ is a submartingale.

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Suppose $(X_t)_{t\geq 0}$ is a martinagle, then

(a) $(|X_t|)_{t\geq 0}$ is a submartingale;

(b) for any $p \geq 1$, $(|X_t|^p)_{t\geq 0}$ is a submartingale if $\mathbb{E}[|X_t|^p] < \infty$ for every $t \geq 0$.

If $(X_t)_{t\geq 0}$ is a submartinagle, then $(X_t^+)_{t\geq 0}$ is a submartinagle.

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Proposition 3.13

Let $(X_t)_{t\geq 0}$ be a submartingale or a supermartingale. Then, for every $t > 0$,

$$\sup_{0\leq s\leq t} \mathbb{E}[|X_s|] < \infty.$$

Proof of Proposition 3.13

We only deal with the case of submartingale. Then (X_t^+) is also a submartingale. Thus, for any $s \in [0, t]$,

$$\mathbb{E}[X_s^+] \leq \mathbb{E}[X_t^+].$$

Since (X_t) is a submartingale, we have, any $s \in [0, t]$,

$$\mathbb{E}[X_s] \geq \mathbb{E}[X_0].$$

Using $|x| = 2x^+ - x$, we get

$$\sup_{0\leq s\leq t} \mathbb{E}[|X_s|] \leq 2\mathbb{E}[X_t^+] - E[X_0] < \infty.$$

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Proposition 3.14

Let (M_t) be a square integrable martingale (i.e., $M_t \in L^2$ for all $t \geq 0$). Let $0 \leq s < t$ and let $s = t_0 < t_1, \cdots < t_p = t$ be a partition of $[s, t]$. Then

$$\mathbb{E}[\sum_{j=1}^{p}(M_{t_j} - M_{t_{j-1}})^2|\mathcal{F}_s] = \mathbb{E}[M_t^2 - M_s^2|\mathcal{F}_s] = \mathbb{E}[(M_t - M_s)^2|\mathcal{F}_s].$$

In particular

$$\mathbb{E}[\sum_{j=1}^{p}(M_{t_j} - M_{t_{j-1}})^2] = \mathbb{E}[M_t^2 - M_s^2] = \mathbb{E}[(M_t - M_s)^2].$$

Proof of Proposition 3.14

For $j = 1, \ldots, p$,

$$\begin{aligned}
\mathbb{E}[(M_{t_j} - M_{t_{j-1}})^2 | \mathcal{F}_s] &= \mathbb{E}[\mathbb{E}[(M_{t_j} - M_{t_{j-1}})^2 | \mathcal{F}_{t_{j-1}} | \mathcal{F}_s] \\
&= \mathbb{E}\left[\mathbb{E}[M_{t_j}^2 | \mathcal{F}_{t_{j-1}}] - 2M_{t_{j-1}} \mathbb{E}[M_{t_j} | \mathcal{F}_{t_{j-1}}] + M_{t_{j-1}}^2 | \mathcal{F}_s\right] \\
&= \mathbb{E}\left[\mathbb{E}[M_{t_j}^2 | \mathcal{F}_{t_{j-1}}] - M_{t_{j-1}}^2 | \mathcal{F}_s\right] \\
&= \mathbb{E}[M_{t_j}^2 - M_{t_{j-1}}^2 | \mathcal{F}_s].
\end{aligned}$$

The following are counter parts of some results for discrete time martingales.

Proposition 3.15

(i) (Maximal inequality) Let $(X_t)_{t\geq 0}$ be a supermartingale with right-continuous sample paths. Then, for every $t > 0$ and every $\lambda > 0$,

$$\lambda \mathbb{P}(\sup_{0\leq s\leq t} |X_s| > \lambda) \leq \mathbb{E}[|X_0|] + 2\mathbb{E}[|X_t|].$$

(ii) (Doob's L^p inequality) Let $(X_t)_{t\geq 0}$ be a martingale with right-continuous sample paths. Then, for every $t > 0$ and every $p > 1$,

$$\mathbb{E}[\sup_{0\leq s\leq t} |X_s|^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X_t|^p].$$

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Proof of Proposition 3.15

(i) Fix $t > 0$ and consider a countable dense subset D of $\mathbb{R}_+$ such that $0 \in D$ and $t \in D$. Then $D \cap [0, t]$ is the increasing union of a sequence $(D_m)_{m \geq 1}$ of finite subsets of $[0, t]$ of the form $D_m = \{t_0^m, t_1^m, \dots, t_m^m\}$ where $0 = t_0^m < t_1^m < \cdots < t_m^m = t$. For every fixed m, we can apply the discrete time maximal inequality to the sequence $Y_n = X_{t_{n \wedge m}}$ (which is a discrete supermartingale with respect to the filtration $\mathcal{G}_n = \mathcal{F}_{t_{n \wedge m}}$) to get

$$\lambda \mathbb{P}(\sup_{s \in D_m} |X_s| > \lambda) \leq \mathbb{E}[|X_0|] + 2\mathbb{E}[|X_t|].$$

Note that as $m \uparrow \infty$,

$$\mathbb{P}(\sup_{s \in D_m} |X_s| > \lambda) \uparrow \mathbb{P}(\sup_{s \in D \cap [0,t]} |X_s| > \lambda).$$

Thus

$$\lambda \mathbb{P}(\sup_{s \in D \cap [0,t]} |X_s| > \lambda) \leq \mathbb{E}[|X_0|] + 2\mathbb{E}[|X_t|].$$

Since (X_t) is right-continuous, we have

$$\sup_{s \in D \cap [0,t]} |X_s| = \sup_{s \in [0,t]} |X_s|. \tag{1}$$

The conclusion of (i) now follows immediately.

(ii) Following the same strategy as in the proof of (i), and using now Doob's L^p inequality for discrete martingales, we get, for every $m \geq 1$

$$\mathbb{E}[\sup_{s \in D_m} |X_s|^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X_t|^p].$$

Now we just have to let m tend to infinity, using the monotone convergence theorem and then the identity (1).

Remark

If we do not assume right continuity, the preceding proof shows that, for every countable dense subset D of $\mathbb{R}_+$, and every $t > 0$,

$$\mathbb{P}(\sup_{s \in D \cap [0,t]} |X_s| > \lambda) \leq \frac{1}{\lambda}(\mathbb{E}[|X_0|] + 2\mathbb{E}[|X_t|]).$$

Letting $\lambda \uparrow \infty$, we get

$$\sup_{s \in D \cap [0,t]} |X_s| < \infty, a.s.$$

Let $f : I \to \mathbb{R}$ be a function defined on a subset I of $\mathbb{R}_+$. If $a < b$, the upcrossing number of f along $[a, b]$, denoted by $M^f_{ab}(I)$, is the maximal integer $k \geq 1$ such that there exists a finite increasing sequence $s_1 < t_1 < \cdots < s_k < t_k$ of elements of I such that $f(s_i) \leq a$ and $f(t_i) \geq b$ for every $i \in \{1, 2, \ldots k\}$. Upcrossing numbers are very useful in studying regularity of functions.

We will use the notation

$$\lim_{s \downarrow \downarrow t} f(s) := \lim_{s \downarrow t, s > t} f(s)$$

and

$$\lim_{s \uparrow \uparrow t} f(s) := \lim_{s \uparrow t, s < t} f(s).$$

We say a function $g : \mathbb{R}_+ \to \mathbb{R}$ is cadlag if g is right continuous and has left limit at every $t > 0$.

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We say a function $g : \mathbb{R}_+ \to \mathbb{R}$ is cadlag if g is right continuous and has left limit at every $t > 0$.

Lemma 3.16

Let D be a countable dense subset of $\mathbb{R}_+$ and let f be a real function defined on D. We assume that, for every $T \in D$,

(i) f is bounded on $D \cap [0, T]$,

(ii) for all rationals a and b such that $a < b$,

$$M^f_{ab}(D \cap [0, T]) < \infty.$$

Then, the right-limit

$$f(t+) := \lim_{s \downarrow\downarrow t, s \in D} f(s)$$

exists for every real $t \geq 0$, and similarly the left-limit

$$f(t-) := \lim_{s \uparrow\uparrow t, s \in D} f(s)$$

exists for every real $t > 0$. Furthermore, the function $g : \mathbb{R}_+ \to \mathbb{R}$ defined by $g(t) = f(t+)$ is cadlag.

Theorem 3.17

Let $(X_t)_{t\geq 0}$ be a supermartingale, and let D be a countable dense subset of $\mathbb{R}_+$.

(i) For almost every $\omega \in \Omega$, the restriction of the function $s \to X_s(\omega)$ to D has a right-limit

$$X_{t+}(\omega) := \lim_{s\downarrow\downarrow t, s\in D} X_s(\omega), \quad \forall t \in [0,\infty),$$

and a left-limit

$$X_{t-}(\omega); = \lim_{s\uparrow\uparrow t, s\in D} X_s(\omega), \quad \forall t \in (0,\infty).$$

(ii) For every $t \geq 0$, $X_{t+} \in L^1$ and

$$X_t \geq \mathbb{E}[X_{t+}|\mathcal{F}_t]$$

with equality if the function $t \mapsto \mathbb{E}[X_t]$ is right-continuous. The process $(X_{t+})_{t\geq 0}$ is a supermartingale with respect to $(\mathcal{F}_{t+})$. It is a martingale if X is a martingale.