Math 562 Fall 2020

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September 16, 2020

Outline

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HW2 is posted on my homepage. I also set up HW2 in the course Moodle page. You need to submit your HW2 via Moodle. The due date for HW2 is 09/18 at noon. Make sure that your HW is uploaded successfully.

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Theorem 3.7

Let $(X_t)_{t\geq 0}$ be a progressive process with values in a measurable space $(E, \mathcal{E})$, and let T be a stopping time. Then the function $\omega \mapsto X_T(\omega) = X_{T(\omega)}(\omega)$, defined on $\{T(\omega) < \infty\}$, is $\mathcal{F}_T$-measurable.

Proof of Theorem 3.7

By property (j), it suffices to show that, for every $t \geq 0$, the restriction of $\omega \mapsto X_T(\omega)$ to $\{T \leq t\}$ is $\mathcal{F}_t$-measurable. This restriction is the composition of

$$\begin{array}{ccc} \{T \leq t\} \ni \omega & \mapsto & (\omega, T(\omega) \wedge t) \\ \mathcal{F}_t & & \mathcal{F}_t \otimes \mathcal{B}([0,t]) \end{array}$$

and

$$\begin{array}{ccc} \Omega \times [0,t] \ni (\omega, s) & \mapsto & X_s(\omega) \\ \mathcal{F}_t \otimes \mathcal{B}([0,t]) & & \mathcal{E} \end{array}$$

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The first map is measurable since $T \wedge t$ is $\mathcal{F}_t$-measurable; and the second one by the definition of a progressive process. The proof is now complete.

Proposition 3.8

Let T be a stopping time and let S be an $\mathcal{F}_T$-measurable random variable with values in $[0, \infty]$, such that $S \geq T$. Then S is also a stopping time.

In particular, if T is a stopping time,

$$T_n = \sum_{k=0}^{\infty} \frac{k+1}{2^n} 1_{\{k \cdot 2^{-n} < T \leq (k+1)2^{-n}\}} + \infty \cdot 1_{\{T=\infty\}}, \quad n = 1, 2, \ldots$$

defines a sequence of stopping times that decreases to T.

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defines a sequence of stopping times that decreases to T.

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For every $t \geq 0$,

$$\{S \leq t\} = \{S \leq t\} \cap \{T \leq t\} \in \mathcal{F}_t$$

since $\{S \leq t\} \in \mathcal{F}_T$. Thus the first assertion is true. The second assertion follows since $T_n \geq T$, T_n, as a function of T, is $\mathcal{F}_T$-measurable, and $T_n \downarrow T$ by construction.

Proposition 3.9

Let $(X_t)_{t\geq 0}$ be an adapted process with values in a metric space (E, d).

(i) Assume that the sample paths of X are right-continuous, and let O be an open subset of E. Then $T_O = \inf\{t \geq 0 : X_t \in O\}$ is a stopping time wrt $(\mathcal{F}_{t+})$.

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Proposition 3.9 (cont)

(ii) Assume that the sample paths of X are continuous, and let F be a closed subset of E. Then $T_F = \inf\{t \geq 0 : X_t \in F\}$ is a stopping time wrt $(\mathcal{F}_t)$.

Proof of Proposition 3.9

(i) For every $t > 0$,

$$\{T_O < t\} = \cup_{s \in [0,t) \cap \mathbb{Q}} \{X_s \in O\} \in \mathcal{F}_t.$$

(ii) For every $t \geq 0$,

$$\{T_F \leq t\} = \{\inf_{s \in [0,t]} d(X_s, F) = 0\} = \{\inf_{s \in [0,t] \cap \mathbb{Q}} d(X_s, F) = 0\} \in \mathcal{F}_t.$$

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Outline

In this section, we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

Definition 3.10

An adapted real-valued process $(X_t)_{t\geq 0}$ such that $X_t \in L^1$ for every $t \geq 0$ is called

- a martingale if, for all $0 \leq s \leq t$, $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$;
- a supermartingale if, for all $0 \leq s \leq t$, $\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s$;
- a submartingale if, for all $0 \leq s \leq t$, $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$.

If $(X_t)_{t\geq 0}$ is a submartingale, then $(-X_t)_{t\geq 0}$ is a supermartingale. Thus some of the results below are stated for supermartingales only, but the analogous results for submartingales immediately follow.

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A simple way to construct a martingale is to take a random variable $Z \in L^1$ and define $X_t = \mathbb{E}[Z|\mathcal{F}_t]$, $t \geq 0$. However, not all martingales are of this type.

We say that a process $(Z_t)_{t\geq 0}$ taking values in $\mathbb{R}$ or $\mathbb{R}^d$ has independent increment wrt $(\mathcal{F}_t)_{t\geq 0}$ if Z is adapted and if, for all $0 \leq s \leq t$, $Z_t - Z_s$ is independent of $\mathcal{F}_s$.

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Suppose $(Z_t)_{t\geq 0}$ is real-valued process with independent increments wrt $(\mathcal{F}_t)_{t\geq 0}$. Then

(i) if $Z_t \in L^1$ for all $t \geq 0$, then $\widetilde{Z}_t = Z_t - \mathbb{E}[Z_t]$ is a martingale;

(ii) if $Z_t \in L^2$ for all $t \geq 0$, then $Y_t = \widetilde{Z}_t^2 - \mathbb{E}[\widetilde{Z}_t^2]$ is a martingale;

(iii) if for some $\theta \in \mathbb{R}$, we have $\mathbb{E}[e^{\theta Z_t}] < \infty$ for all $t \geq 0$, then

$$X_t = \frac{e^{\theta Z_t}}{\mathbb{E}[e^{\theta Z_t}]}$$

is a martingale

(i) For all $0 \leq s \leq t$,

$$\mathbb{E}[\widetilde{Z}_t|\mathcal{F}_s] = \mathbb{E}[\widetilde{Z}_s + (Z_t - Z_s - \mathbb{E}[Z_t - Z_s])|\mathcal{F}_s] = \widetilde{Z}_s.$$

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$$\begin{aligned}
\mathbb{E}[\widetilde{Z}_t^2|\mathcal{F}_s] &= \mathbb{E}[(\widetilde{Z}_s + (\widetilde{Z}_t - \widetilde{Z}_s))^2|\mathcal{F}_s] \\
&= \widetilde{Z}_s^2 + 2\widetilde{Z}_s\mathbb{E}[\widetilde{Z}_t - \widetilde{Z}_s|\mathcal{F}_s] + \mathbb{E}[(\widetilde{Z}_t - \widetilde{Z}_s)^2|\mathcal{F}_s] \\
&= \widetilde{Z}_s^2 + \mathbb{E}[(\widetilde{Z}_t - \widetilde{Z}_s)^2] \\
&= \widetilde{Z}_s^2 + \mathbb{E}[\widetilde{Z}_t^2] - 2\mathbb{E}[\widetilde{Z}_t\widetilde{Z}_s] + \mathbb{E}[\widetilde{Z}_s^2] \\
&= \widetilde{Z}_s^2 + \mathbb{E}[\widetilde{Z}_t^2] - \mathbb{E}[\widetilde{Z}_s^2]
\end{aligned}$$

since $\mathbb{E}[\widetilde{Z}_t\widetilde{Z}_s] = \mathbb{E}[\widetilde{Z}_s\mathbb{E}[\widetilde{Z}_t|\mathcal{F}_s]] = \mathbb{E}[\widetilde{Z}_s^2]$.
(iii) For all $0 \leq s \leq t$,

$$\mathbb{E}[X_t|\mathcal{F}_s] = \frac{e^{\theta Z_s}\mathbb{E}[e^{\theta(Z_t - Z_s)}|\mathcal{F}_s]}{\mathbb{E}[e^{\theta Z_s}]\mathbb{E}[e^{\theta(Z_t - Z_s)}]} = \frac{e^{\theta Z_s}}{\mathbb{E}[e^{\theta Z_s}]} = X_s.$$

Definition 3.11

A real-valued process $B = (B_t)_{t\geq 0}$ is said to be an $(\mathcal{F}_t)$-Brownian motion if B is a Brownian motion and if B is adapted and has independent increments wrt $(\mathcal{F}_t)$. Similarly, a process $B = (B_t)_{t\geq 0}$ with values in $\mathbb{R}^d$ is said to be a d-dim $(\mathcal{F}_t)$-Brownian motion if B is a d-dim Brownian motion and if B is adapted and has independent increments wrt $(\mathcal{F}_t)$.

If B is a (d-dimensional) Brownian motion and $(\mathcal{F}_t^B)$ is the (possibly completed) canonical filtration of B, then B is a (d-dimensional) $(\mathcal{F}_t^B)$-Brownian motion.

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If B is a (d-dimensional) Brownian motion and $(\mathcal{F}_t^B)$ is the (possibly completed) canonical filtration of B, then B is a (d-dimensional) $(\mathcal{F}_t^B)$-Brownian motion.

Let B be an $(\mathcal{F}_t)$-Brownian motion started from 0 (or from $a \in \mathbb{R}$), then

$$B_t, \quad B_t^2 - t, \quad e^{\theta B_t - \frac{\theta^2}{2} t}$$

are martingales with continuous sample paths. The processes $e^{\theta B_t - \frac{\theta^2}{2} t}$ are called exponential martingales of Brownian motion.

For $f \in L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dt)$, define

$$Z_t = \int_0^t f(s) dB_s.$$

Z has independent increments with respect to the canonical filtration of B.

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Thus

$$\int_0^t f(s)dB_s, \quad \left(\int_0^t f(s)dB_s\right)^2 - \int_0^t f^2(s)ds,$$

and

$$\exp\left(\theta \int_0^t f(s)dB_s - \frac{\theta^2}{2}\int_0^t f^2(s)ds\right)$$

are martingales. One can prove that these martingales have a modification with continuous sample paths.

If $Z = (N_t)_{t\geq 0}$ is a Poisson process with parameter λ wit associated canonical filtration $(\mathcal{F}_t)$, then Z has independent increments wrt $(\mathcal{F}_t)$. Thus

$$N_t - \lambda t, \quad (N_t - \lambda t)^2 - \lambda t, \quad \exp\left(\theta N_t - \lambda t(e^\theta - 1)\right)$$

are martingales. These martingales do not have a modification with continuous sample paths.

proposition 3.1

Let $(X_t)_{t\geq 0}$ be an adapted process and let $f : \mathbb{R} \to \mathbb{R}$ be a convex function such that $\mathbb{E}[|f(X_t)|] < \infty$ for all $t \geq 0$.

(i) If $(X_t)_{t\geq 0}$ is a martinagle, then $(f(X_t))$ is a submartingale;

(ii) If $(X_t)_{t\geq 0}$ is a submartinagle, and if in addition f is non-decreasing, then $(f(X_t))$ is a submartingale.

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Suppose $(X_t)_{t\geq 0}$ is a martinagle, then

(a) $(|X_t|)_{t\geq 0}$ is a submartingale;

(b) for any $p \geq 1$, $(|X_t|^p)_{t\geq 0}$ is a submartingale if $\mathbb{E}[|X_t|] < \infty$ for every $t \geq o$.

If $(X_t)_{t\geq 0}$ is a submartinagle, then $(X_t^+)_{t\geq 0}$ is a submartinagle.

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