Math 562 Fall 2020

Renming Song

University of Illinois at Urbana-Champaign

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Outline

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HW2 is posted on my homepage. I also set up HW2 in the course Moodle page. You need to submit your HW2 via Moodle. The due date for HW1 is 09/18 at noon. Make sure that your HW is uploaded successfully.

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Throughout this chapter, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 3.2

A process $X = (X_t)_{t \geq 0}$ taking values in a measurable space $(E, \mathcal{E})$ is said to be measurable if the mapping

$$\Omega \times [0, \infty) \ni (\omega, t) \mapsto X_t(\omega) \in E$$

is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.

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In the remaining part of this chapter, we fix a filtration $(\mathcal{F}_t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$, and the notions that will be introduced depend on the choice of this filtration.

Definition 3.3

A process $X = (X_t)_{t\geq 0}$ taking values in a measurable space $(E, \mathcal{E})$ is said to be adapted if, for every $t \geq 0$, X_t is $\mathcal{F}_t$-measurable.
$X = (X_t)_{t\geq 0}$ is said to be progressively measurable if, for every $t \geq 0$, the mapping

$$\Omega \times [0, t] \ni (\omega, s) \mapsto X_t(\omega) \in E$$

is $\mathcal{F}_t \otimes \mathcal{B}([0, t])$-measurable.

Progressive measurability implies adaptedness and measurability.

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Progressive measurability implies adaptedness and measurability.

Proposition 3.4

Let $X = (X_t)_{t\geq 0}$ be a process taking values in a metric space (E, d). Suppose that X is adapted and that the sample paths of X are right-continuous (i.e., for every $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is right-continuous. Then X is progressively measurable. The same conclusion holds if one replaces right-continuity by left-continuity.

Proof of Proposition 3.4

Fix $t > 0$. For every $n \geq 1$ and $s \in [0, t]$, define

$$X_s^n = X_{(kt)/n} \quad \text{if } s \in [\frac{(k-1)t}{n}, \frac{kt}{n}), k \in \{1, \dots, n\}$$

and $X_t^n = X_t$. Then by right-continuity, for any $s \in [0, t]$ and any $\omega \in \Omega$,

$$X_s(\omega) = \lim_{n\to\infty} X_s^n(\omega).$$

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and $X_t^n = X_t$. Then by right-continuity, for any $s \in [0, t]$ and any $\omega \in \Omega$,

$$X_s(\omega) = \lim_{n \to \infty} X_s^n(\omega).$$

Proof of Proposition 3.4

For any $A \in \mathcal{B}(E)$,

$$\{(\omega, s) \in \Omega \times [0, t] : X_s^n(\omega) \in A\}$$
$$= (\{X_t \in A\} \times \{t\}) \cup \left(\cup_{k=1}^n \{X_{(kt)/n} \in A\} \times \left[\frac{(k-1)t}{n}, \frac{kt}{n}\right)\right)$$
$$\in \mathcal{F}_t \otimes \mathcal{B}([0, t]).$$

The collection $\mathcal{P}$ of all sets $A \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ such that the process $X_t(\omega) = 1_A(\omega, t)$ is progressively measurable is a σ-field on $\Omega \times \mathbb{R}_+$, and it is called the progressive σ-field.

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A subset $A \subset \Omega \times \mathbb{R}_+$ belongs to $\mathcal{P}$ if and only if, for any $t \geq 0$, $A \cap (\Omega \times [0, t]) \in \mathcal{F}_t \otimes \mathcal{B}([0, t])$.

A process X is progressively measurable if and only if X is measurable with respect to $\mathcal{P}$.

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Definition 3.5

A random variable $T : \Omega \to [0, \infty]$ is said to be a stopping time with respect to $(\mathcal{F}_t)$ if, for any $t \geq 0$, $\{T \leq t\} \in \mathcal{F}_t$.

The σ-field of the past before T is

$$\mathcal{F}_T = \{A \in \mathcal{F}_\infty : \forall t \geq 0, A \cap \{T \leq t\} \in \mathcal{F}_t\}.$$

If T a stopping time with respect to $(\mathcal{F}_t)$, then for any $t > 0$,

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If T a stopping time with respect to $(\mathcal{F}_t)$, then for any $t > 0$,

$$\{T < t\} \in \mathcal{F}_t.$$

Proposition 3.6

Let $\mathcal{G}_t = \mathcal{F}_{t+}$ for every $t \in [0, \infty]$.

(i) A random variable $T : \Omega \to [0, \infty]$ is a stopping time respect to $(\mathcal{G}_t)$ if and only if , for every $t > 0$, $\{T < t\} \in \mathcal{F}_t$. This is also equivalent to saying that $T \wedge t$ is $\mathcal{F}_t$-measurable for every $t > 0$.

(ii) Let T be a stopping time respect to $(\mathcal{G}_t)$. Then

$$\mathcal{G}_T = \{A \in \mathcal{F}_\infty : \forall t > 0 : A \cap \{T < t\} \in \mathcal{F}_t\}.$$

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$$\mathcal{F}_{T+} := \mathcal{G}_T.$$

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(ii) Let T be a stopping time respect to $(\mathcal{G}_t)$. Then

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We will write

$$\mathcal{F}_{T+} := \mathcal{G}_T.$$

Proof of Proposition 3.6

(i) Suppose that T is a stopping time wrt $(\mathcal{G}_t)$. Then, for every $t > 0$,

$$\{T < t\} = \cup_{\mathbb{Q}_+ \ni q < t}\{T \leq q\} \in \mathcal{F}_t$$

since $\{T \leq q\} \in \mathcal{G}_q \subset \mathcal{F}_t$ if $q < t$. Conversely, assume that $\{T < t\} \in \mathcal{F}_t$ for every $t > 0$. Then for every $t \geq 0$ and $s > t$,

$$\{T \leq t\} = \cap_{q \in \mathbb{Q}_+, t < q < s}\{T < q\} \in \mathcal{F}_s$$

and it follows that $\{T \leq t\} \in \mathcal{F}_{t+} = \mathcal{G}_t$.

$T \wedge t$ is $\mathcal{F}_t$-measurable for every $t > 0$ is equivalent to saying that, for every $s < t$, $\{T \leq s\} \in \mathcal{F}_t$. Taking a sequence of values of s that increases to t, we see that the latter property implies that $\{T < t\} \in \mathcal{F}_t$, and so T is a stopping time wrt $(\mathcal{G}_t)$. Conversely, if T is a stopping time wrt $(\mathcal{G}_t)$, we have $\{T \leq s\} \in \mathcal{G}_s \subset \mathcal{F}_t$ whenever $s < t$, and thus $T \wedge t$ is $\mathcal{F}_t$-measurable.

Proof of Proposition 3.6 (cont)

(ii) First, if $A \in \mathcal{G}_T$, we have $A \cap \{T \leq t\} \in \mathcal{G}_t$ for every $t \geq 0$. Hence, for $t > 0$,

$$A \cap \{T < t\} = \cup_{\mathbb{Q}_+ \ni q < t} (A \cap \{T \leq q\}) \in \mathcal{F}_t$$

since $A \cap \{T \leq q\} \in \mathcal{G}_q \subset \mathcal{F}_t$ whenever $q < t$.

Conversely, assume that $A \cap \{T < t\} \in \mathcal{F}_t$ for every $t > 0$. Then, for every $t \geq 0$ and $s > t$,

$$A \cap \{T \leq t\} = \cap_{q \in \mathbb{Q}_+, t < q < s} (A \cap \{T < q\}) \in \mathcal{F}_s.$$

Thus, $A \cap \{T \leq t\} \in \mathcal{F}_{t+} = \mathcal{G}_t$ and hence $A \in \mathcal{G}_T$.

Properties of stopping times and of the associated σ-fields

(a) For any stopping time T, $\mathcal{F}_T \subset \mathcal{F}_{T+}$. If $(\mathcal{F}_t)$ is right-continuous, then $\mathcal{F}_T = \mathcal{F}_{T+}$.

(b) If $T = t$ is a constant stopping time, then $\mathcal{F}_T = \mathcal{F}_t$ and $\mathcal{F}_{T+} = \mathcal{F}_{t+}$.

(c) For any stopping time T, T is $\mathcal{F}_T$-measurable.

(d) Let T be a stopping time and $A \in \mathcal{F}_\infty$. Set

$$T^A(\omega) = \begin{cases} T(\omega) \text{ if } \omega \in A, \\ \infty \text{ if } \omega \notin A. \end{cases} \quad .$$

Then $A \in \mathcal{F}_T$ if and only if T^A is a stopping time.

(e) Let S, T be two stopping times such that $S \leq T$. Then $\mathcal{F}_S \subset \mathcal{F}_T$ and $\mathcal{F}_{S+} \subset \mathcal{F}_{T+}$.

(f) Let S, T be two stopping times. Then $S \vee T$ and $S \wedge T$ are also stopping times and $\mathcal{F}_{S\wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$. Furthermore, $\{S \leq T\} \in \mathcal{F}_{S\wedge T}$ and $\{S = T\} \in \mathcal{F}_{S\wedge T}$.

(g) If (S_n) is a monotone increasing sequence of stopping times, then $S = \lim \uparrow S_n$ is also a stopping time.

(h) If (S_n) is a monotone decreasing sequence of stopping times, then $S = \lim \downarrow S_n$ is a stopping time wrt $(\mathcal{F}_{t+})$ and

$$\mathcal{F}_{S+} = \cap_n \mathcal{F}_{S_n+}.$$

(i) If (S_n) is a monotone decreasing sequence of stopping times, which is also stationary (in the sense that, for every ω, there exists an integer $N(\omega)$ such that $S_n(\omega) = S(\omega)$ for every $n \geq N(\omega)$) then $S = \lim \downarrow S_n$ is a stopping time and

$$\mathcal{F}_S = \cap_n \mathcal{F}_{S_n}.$$

(j) Let T be a stopping time. A function $\omega \mapsto Y(\omega)$ defined on $\{T(\omega) < \infty\}$ and taking values in the measurable space $(E, \mathcal{E})$ is $\mathcal{F}_T$-measurable if and only if, for every $t \geq 0$, $Y|_{\{T \leq t\}}$ is $\mathcal{F}_t$-measurable.

Proof

(a), (b) and (c) are trivial. So we start with
(d) For every $t \geq 0$,

$$\{T^A \leq t\} = A \cap \{T \leq t\}$$

and the result follows from the definition of $\mathcal{F}_T$.
(e) It suffices to prove that $\mathcal{F}_S \subset \mathcal{F}_T$. If $A \in \mathcal{F}_S$, we have

$$A \cap \{T \leq t\} = (A \cap \{S \leq t\}) \cap \{T \leq t\} \in \mathcal{F}_t,$$

and hence $A \in \mathcal{F}_T$.

Proof

(f) We have

$$\{S \wedge T \le t\} = \{S \le t\} \cup \{T \le t\} \in \mathcal{F}_t,$$
$$\{S \vee T \le t\} = \{S \le t\} \cap \{T \le t\} \in \mathcal{F}_t,$$

and so $S \wedge T$ and $S \vee T$ are stopping times.

It follows from (e) that $\mathcal{F}_{S \wedge T} \subset \mathcal{F}_S \cap \mathcal{F}_T$. If $A \in \mathcal{F}_S \cap \mathcal{F}_T$,

$$A \cap \{S \wedge T \le t\} = (A \cap \{S \le t\}) \cup (A \cap \{T \le t\}) \in \mathcal{F}_t,$$

and so $A \in \mathcal{F}_{S \wedge T}$.

For every $t \ge 0$,

$$\{S \le T\} \cap \{T \le t\} = \{S \le t\} \cap \{T \le t\} \cap \{S \wedge t \le T \wedge t\} \in \mathcal{F}_t$$
$$\{S \le T\} \cap \{S \le t\} = \{S \wedge t \le T \wedge t\} \cap \{S \le t\} \in \mathcal{F}_t$$

since $S \wedge t$ and $T \wedge t$ are $\mathcal{F}_t$-measurable (Prop 3.6 (i)). Thus $\{S \le T\} \in \mathcal{F}_S \cap \mathcal{F}_T = \mathcal{F}_{S \wedge T}$. Then $\{S = T\} = \{S \le T\} \cap \{T \le S\}$.

Proof

(g) For every $t \geq 0$,

$$\{S \leq t\} = \cap_n \{S_n \leq t\} \in \mathcal{F}_t.$$

(h) For every $t > 0$,

$$\{S < t\} = \cup_n \{S_n < t\} \in \mathcal{F}_t,$$

and we use Proposition 3.6 (i). Then, by (e), we have $\mathcal{F}_{S+} \subset \mathcal{F}_{S_n+}$ for every n, and conversely, if $A \in \cap_n \mathcal{F}_{S_n+}$,

$$A \cap \{S < t\} = \cup_n (A \cap \{S_n < t\}) \in \mathcal{F}_t,$$

and hence $A \in \mathcal{F}_{S+}$.

Proof

(i) For every $t \geq 0$,

$$\{S \leq t\} = \cup_n \{S_n \leq t\} \in \mathcal{F}_t,$$

and if $A \in \cap_n \mathcal{F}_{S_n}$,

$$A \cap \{S \leq t\} = \cup_n (A \cap \{S_n \leq t\}) \in \mathcal{F}_t,$$

so that $A \in \mathcal{F}_S$.

Proof

(j) First assume that, for every $t \geq 0$, $Y|_{\{T \leq t\}}$ is $\mathcal{F}_t$-measurable. Then for every $A \in \mathcal{E}$,

$$\{Y \in A\} \cap \{T \leq t\} \in \mathcal{F}_t.$$

Letting $t \uparrow \infty$, we first obtain $A \in \mathcal{F}_\infty$, and then we deduce from the previous display that $\{Y \in\} \in \mathcal{F}_T$.

Conversely, if Y is $\mathcal{F}_T$-measurable, $\{Y \in\} \in \mathcal{F}_T$ and thus

$$\{Y \in A\} \cap \{T \leq t\} \in \mathcal{F}_t,$$

giving the desired result.