Math 562 Fall 2020

Renming Song

University of Illinois at Urbana-Champaign

September 11, 2020

Outline

Outline

HW2 is posted on my homepage. I also setup HW2 in the course Moodle page. You need to submit your HW2 via Moodle. The due date for HW1 is 09/18 at noon.

Slides and videos of the lectures are also available from the Moodle page.

HW2 is posted on my homepage. I also setup HW2 in the course Moodle page. You need to submit your HW2 via Moodle. The due date for HW1 is 09/18 at noon.

Slides and videos of the lectures are also available from the Moodle page.

Outline

Theorem 2.20 (Strong Markov Property)

Let T be a stopping time. We assume that $\mathbb{P}(T < \infty) > 0$ and define, for every $t \geq 0$,

$$B_t^{(T)} = 1_{\{T<\infty\}}(B_{T+t} - B_T).$$

Then under the probability measure $\mathbb{P}(\cdot | T < \infty)$, the process $(B_t^{(T)})_{t\geq 0}$ is a Brownian motion independent of $\mathcal{F}_T$.

An important application of the strong Markov property is the“reflection principle” that leads to the following theorem.

Theorem 2.20 (Strong Markov Property)

Let T be a stopping time. We assume that $\mathbb{P}(T < \infty) > 0$ and define, for every $t \geq 0$,

$$B_t^{(T)} = 1_{\{T<\infty\}}(B_{T+t} - B_T).$$

Then under the probability measure $\mathbb{P}(\cdot | T < \infty)$, the process $(B_t^{(T)})_{t\geq 0}$ is a Brownian motion independent of $\mathcal{F}_T$.

An important application of the strong Markov property is the“reflection principle” that leads to the following theorem.

Theorem 2.21

For $t > 0$, set $S_t = \sup_{s \leq t} B_s$. Then, if $a \geq 0$ and $b \in (-\infty, a]$, we have

$$\mathbb{P}(S_t \geq a, B_t \leq b) = \mathbb{P}(B_t \geq 2a - b).$$

In particular, S_t has the same distribution as $|B_t|$.

Proof of Theorem 2.21

We apply the strong Markov property at the stopping time

$$T_a = \inf\{t \geq 0 : B_t = a\}.$$

Note that $\mathbb{P}(T_a < \infty) = 1$.

Theorem 2.21

For $t > 0$, set $S_t = \sup_{s \leq t} B_s$. Then, if $a \geq 0$ and $b \in (-\infty, a]$, we have

$$\mathbb{P}(S_t \geq a, B_t \leq b) = \mathbb{P}(B_t \geq 2a - b).$$

In particular, S_t has the same distribution as $|B_t|$.

Proof of Theorem 2.21

We apply the strong Markov property at the stopping time

$$T_a = \inf\{t \geq 0 : B_t = a\}.$$

Note that $\mathbb{P}(T_a < \infty) = 1$.

Proof of Theorem 2.21 (cont)

Using the notation of Theorem 2.20, we have

$$\mathbb{P}(S_t \geq a, B_t \leq b) = \mathbb{P}(T_a \leq t, B_t \leq b) = \mathbb{P}(T_a \leq t, B^{(T_a)}_{t-T_a} \leq b - a)$$

since $B^{(T_a)}_{t-T_a} = B_t - B_{T_a} = B_t - a$ on the event $\{T_a \leq t\}$. Write $B' = B^{(T_a)}$, then by the strong Markov property, B' is a Brownian motion independent of $\mathcal{F}_{T_a}$ and hence independent of T_a. Since B' has the same law as $-B'$, the pair (T_a, B') also has the same law as $(T_a, -B')$. Let

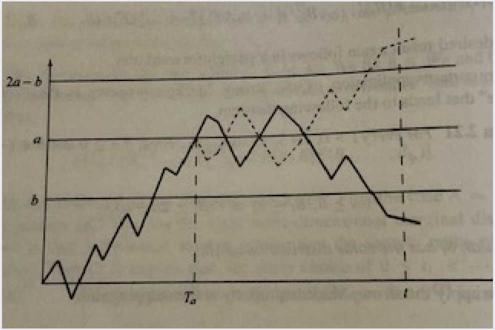
$$H = \{(s, w) \in \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}) : s \leq t, w(t - s) \leq b - a\}.$$

Then $\mathbb{P}(T_a \leq t, B^{(T_a)}_{t-T_a} \leq b - a)$ is equal to

Proof of Theorem 2.21 (cont)

$$\begin{aligned}
&\mathbb{P}((T_a, B') \in H)) = \mathbb{P}((T_a, -B') \in H) \\
&= \mathbb{P}(T_a \leq t, -B^{(T_a)}_{t-T_a} \leq b - a) = \mathbb{P}(T_a \leq t, B_t \geq 2a - b) \\
&= \mathbb{P}(B_t \geq 2a - b)
\end{aligned}$$

since $\{B_t \geq 2a - b\} \subset \{T_a \leq t\}$. This gives the first assertion.



Proof of Theorem 2.21 (cont)

For the last assertion of the theorem, we observe that

$$\begin{aligned}\mathbb{P}(S_t \geq a) &= \mathbb{P}(S_t \geq a, B_t \geq a) + \mathbb{P}(S_t \geq a, B_t \leq a) \\ &= 2\mathbb{P}(B_t \geq a) = \mathbb{P}(|B_t| \geq a)\end{aligned}$$

and the desired result follows.

It follows from the previous theorem that the law (S_t, B_t) has density

$$g(a,b) = \frac{2(2a-b)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2a-b)^2}{2t}\right) 1_{\{a>0, b<a\}}.$$

Proof of Theorem 2.21 (cont)

For the last assertion of the theorem, we observe that

$$\begin{aligned}\mathbb{P}(S_t \geq a) &= \mathbb{P}(S_t \geq a, B_t \geq a) + \mathbb{P}(S_t \geq a, B_t \leq a) \\ &= 2\mathbb{P}(B_t \geq a) = \mathbb{P}(|B_t| \geq a)\end{aligned}$$

and the desired result follows.

It follows from the previous theorem that the law (S_t, B_t) has density

$$g(a,b) = \frac{2(2a-b)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2a-b)^2}{2t}\right) 1_{\{a>0, b<a\}}.$$

Corollary 2.22

For any $a > 0$, T_a has the same distribution as $\frac{a^2}{B_1^2}$ and has density

$$f(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right) 1_{\{a>0\}}.$$

Consequently $\mathbb{E}[T_a] = \infty$.

Proof of Corollary 2.22

For any $t \geq 0$,

$$\mathbb{P}(T_a \leq t) = \mathbb{P}(S_t \geq a) = \mathbb{P}(|B_t| \geq a) = P(B_t^2 \geq a^2)$$
$$= \mathbb{P}(tB_1^2 \geq a^2) = \mathbb{P}(\frac{a^2}{B_1^2} \leq t).$$

A straight forward calculation gives the density of $\frac{a^2}{B_1^2}$.

Corollary 2.22

For any $a > 0$, T_a has the same distribution as $\frac{a^2}{B_1^2}$ and has density

$$f(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right) 1_{\{a>0\}}.$$

Consequently $\mathbb{E}[T_a] = \infty$.

Proof of Corollary 2.22

For any $t \geq 0$,

$$\mathbb{P}(T_a \leq t) = \mathbb{P}(S_t \geq a) = \mathbb{P}(|B_t| \geq a) = P(B_t^2 \geq a^2)$$
$$= \mathbb{P}(tB_1^2 \geq a^2) = \mathbb{P}(\frac{a^2}{B_1^2} \leq t).$$

A straight forward calculation gives the density of $\frac{a^2}{B_1^2}$.

Definition 2.23

If Z is a real-valued random variable, a process $(X_t)_{t\geq 0}$ is said to be a real-valued Brownian motion started from Z if we can write $X_t = Z + B_t$, where B is a real-valued Brownian motion started from 0 and is independent of Z.

Definition 2.24

A process $B_t = (B_t^1, \ldots, B_t^d)$ with values in $\mathbb{R}^d$ is a d-dimensional Brownian motion started from 0 if its components $B^1, \ldots, B^d$ are independent real-valued Brownian motions started from 0. If Z is an $\mathbb{R}^d$-valued random variable and $X_t = (X_t^1, \ldots, X_t^d)$ is a process with values in $\mathbb{R}^d$, we say that X is a d-dimensional Brownian motion started from Z if we can write $X_t = Z + B_t$, where B is a d-dimen Brownian motion started from 0 and is independent of Z.

Definition 2.23

If Z is a real-valued random variable, a process $(X_t)_{t \geq 0}$ is said to be a real-valued Brownian motion started from Z if we can write $X_t = Z + B_t$, where B is a real-valued Brownian motion started from 0 and is independent of Z.

Definition 2.24

A process $B_t = (B_t^1, \dots, B_t^d)$ with values in $\mathbb{R}^d$ is a d-dimensional Brownian motion started from 0 if its components $B^1, \dots, B^d$ are independent real-valued Brownian motions started from 0. If Z is an $\mathbb{R}^d$-valued random variable and $X_t = (X_t^1, \dots, X_t^d)$ is a process with values in $\mathbb{R}^d$, we say that X is a d-dimensional Brownian motion started from Z if we can write $X_t = Z + B_t$, where B is a d-dimen Brownian motion started from 0 and is independent of Z.

Note that, if X is a d-dimensional Brownian motion and the initial value of X is random, the components of X may not be independent because the initial value may introduce some dependence (this does not occur if the initial value is deterministic).

Let $C(\mathbb{R}_+, \mathbb{R}^d)$ be the space of continuous maps from $\mathbb{R}_+$ to $\mathbb{R}^d$. We can equip it with the metric as in the case $d = 1$. Let $\mathcal{C}$ be the Borel σ-field on $C(\mathbb{R}_+, \mathbb{R}^d)$. The d-dimensional Wiener measure is defined as the probability measure on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{C})$ which is the law of a d-dimensional Brownian motion started from 0. The canonical construction of Sect. 2.2 also applies to d-dimensional Brownian motion.

Note that, if X is a d-dimensional Brownian motion and the initial value of X is random, the components of X may not be independent because the initial value may introduce some dependence (this does not occur if the initial value is deterministic).

Let $C(\mathbb{R}_+, \mathbb{R}^d)$ be the space of continuous maps from $\mathbb{R}_+$ to $\mathbb{R}^d$. We can equip it with the metric as in the case $d = 1$. Let $\mathcal{C}$ be the Borel σ-field on $C(\mathbb{R}_+, \mathbb{R}^d)$. The d-dimensional Wiener measure is defined as the probability measure on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{C})$ which is the law of a d-dimensional Brownian motion started from 0. The canonical construction of Sect. 2.2 also applies to d-dimensional Brownian motion.

Many of the results about real-valued Brownian motions started from 0 can be extended to d-dimensional Brownian motion with an arbitrary starting point. In particular, the invariance properties of Proposition 2.5 still hold with the obvious adaptations. Furthermore, property (i) of Proposition 2.5 can be extended as follows: If X is a d-dimensional Brownian motion and Φ is an isometry of $\mathbb{R}^d$, the process $(\Phi(X_t))_{t\geq 0}$ is still a d-dimensional Brownian motion.

Blumenthal's zero-one law can be easily extended, and the strong Markov property also holds.

Many of the results about real-valued Brownian motions started from 0 can be extended to d-dimensional Brownian motion with an arbitrary starting point. In particular, the invariance properties of Proposition 2.5 still hold with the obvious adaptations. Furthermore, property (i) of Proposition 2.5 can be extended as follows: If X is a d-dimensional Brownian motion and Φ is an isometry of $\mathbb{R}^d$, the process $(\Phi(X_t))_{t\geq 0}$ is still a d-dimensional Brownian motion.

Blumenthal's zero-one law can be easily extended, and the strong Markov property also holds.

Outline

Throughout this chapter, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 3.1

A filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection $(\mathcal{F}_t)_{0 \le t \le \infty}$ indexed by $[0, \infty]$ of sub-σ-fields of $\mathcal{F}$, such that $\mathcal{F}_s \subset \mathcal{F}_t$ for all $0 \le s \le t \le \infty$.

If $(\mathcal{F}_t)_{0 \le t \le \infty}$ is a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$, then for all $0 \le s \le t \le \infty$,

$$\mathcal{F}_0 \subset \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_\infty \subset \mathcal{F}.$$

$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is called a filtered probability space.

Throughout this chapter, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 3.1

A filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection $(\mathcal{F}_t)_{0 \le t \le \infty}$ indexed by $[0, \infty]$ of sub-σ-fields of $\mathcal{F}$, such that $\mathcal{F}_s \subset \mathcal{F}_t$ for all $0 \le s \le t \le \infty$.

If $(\mathcal{F}_t)_{0 \le t \le \infty}$ is a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$, then for all $0 \le s \le t \le \infty$,

$$\mathcal{F}_0 \subset \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_\infty \subset \mathcal{F}.$$

$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is called a filtered probability space.

Throughout this chapter, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 3.1

A filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection $(\mathcal{F}_t)_{0 \leq t \leq \infty}$ indexed by $[0, \infty]$ of sub-σ-fields of $\mathcal{F}$, such that $\mathcal{F}_s \subset \mathcal{F}_t$ for all $0 \leq s \leq t \leq \infty$.

If $(\mathcal{F}_t)_{0 \leq t \leq \infty}$ is a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$, then for all $0 \leq s \leq t \leq \infty$,

$$\mathcal{F}_0 \subset \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_\infty \subset \mathcal{F}.$$

$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is called a filtered probability space.

Example

If B is a Brownian motion, then

$$\mathcal{F}_t = \sigma(B_s : 0 \leq s \leq t), \quad \mathcal{F}_\infty = \sigma(B_s : s \geq 0)$$

is the canonical (natural) filtration of B.

More generally, if $X = (X_t)_{t \geq 0}$ is any process, then the canonical (natural) filtration of X is defined by

$$\mathcal{F}_t^X = \sigma(X_s : 0 \leq s \leq t), \quad \mathcal{F}_\infty^X = \sigma(X_s : s \geq 0).$$

If $(\mathcal{F}_t)_{0 \leq t \leq \infty}$ is a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$, define

$$\mathcal{F}_{t+} = \cap_{s>t} \mathcal{F}_s, \quad \mathcal{F}_{\infty+} = \mathcal{F}_\infty.$$

Then $(\mathcal{F}_{t+})_{0 \leq t \leq \infty}$ is also a filtration.

Example

If B is a Brownian motion, then

$$\mathcal{F}_t = \sigma(B_s : 0 \le s \le t), \quad \mathcal{F}_\infty = \sigma(B_s : s \ge 0)$$

is the canonical (natural) filtration of B.

More generally, if $X = (X_t)_{t\ge0}$ is any process, then the canonical (natural) filtration of X is defined by

$$\mathcal{F}_t^X = \sigma(X_s : 0 \le s \le t), \quad \mathcal{F}_\infty^X = \sigma(X_s : s \ge 0).$$

If $(\mathcal{F}_t)_{0\le t\le\infty}$ is a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$, define

$$\mathcal{F}_{t+} = \cap_{s>t}\mathcal{F}_s, \quad \mathcal{F}_{\infty+} = \mathcal{F}_\infty.$$

Then $(\mathcal{F}_{t+})_{0\le t\le\infty}$ is also a filtration.

Example

If B is a Brownian motion, then

$$\mathcal{F}_t = \sigma(B_s : 0 \leq s \leq t), \quad \mathcal{F}_\infty = \sigma(B_s : s \geq 0)$$

is the canonical (natural) filtration of B.

More generally, if $X = (X_t)_{t \geq 0}$ is any process, then the canonical (natural) filtration of X is defined by

$$\mathcal{F}_t^X = \sigma(X_s : 0 \leq s \leq t), \quad \mathcal{F}_\infty^X = \sigma(X_s : s \geq 0).$$

If $(\mathcal{F}_t)_{0 \leq t \leq \infty}$ is a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$, define

$$\mathcal{F}_{t+} = \cap_{s>t} \mathcal{F}_s, \quad \mathcal{F}_{\infty+} = \mathcal{F}_\infty.$$

Then $(\mathcal{F}_{t+})_{0 \leq t \leq \infty}$ is also a filtration.

We say that a filtration $(\mathcal{F}_t)_{0\le t\le\infty}$ is right-continuous if

$$\mathcal{F}_{t+} = \mathcal{F}_t, \quad \text{for all } t \ge 0.$$

Let $(\mathcal{F}_t)_{0\le t\le\infty}$ be a filtration and let $\mathcal{N}$ be the collection of all $(\mathcal{F}_\infty, \mathbb{P})$-negligible sets ($A \in \mathcal{N}$ is there exists $A' \in \mathcal{F}_\infty$ such that $A \subset A'$ and $\mathbb{P}(A') = 0$). The filtration $(\mathcal{F}_t)_{0\le t\le\infty}$ is said to be complete if $\mathcal{N} \subset \mathcal{F}_0$.

If $(\mathcal{F}_t)_{0\le t\le\infty}$ is not complete, it can be completed by setting $\mathcal{F}'_t = \mathcal{F}_t \vee \sigma(\mathcal{N})$. We will often apply this completion procedure to the canonical filtration of a process X and call the resulting filtration the completed canonical filtration of X.

We say that a filtration $(\mathcal{F}_t)_{0\leq t\leq\infty}$ is right-continuous if

$$\mathcal{F}_{t+} = \mathcal{F}_t, \quad \text{for all } t \geq 0.$$

Let $(\mathcal{F}_t)_{0\leq t\leq\infty}$ be a filtration and let $\mathcal{N}$ be the collection of all $(\mathcal{F}_\infty, \mathbb{P})$-negligible sets ($A \in \mathcal{N}$ is there exists $A' \in \mathcal{F}_\infty$ such that $A \subset A'$ and $\mathbb{P}(A') = 0$). The filtration $(\mathcal{F}_t)_{0\leq t\leq\infty}$ is said to be complete if $\mathcal{N} \subset \mathcal{F}_0$.

If $(\mathcal{F}_t)_{0\leq t\leq\infty}$ is not complete, it can be completed by setting $\mathcal{F}'_t = \mathcal{F}_t \vee \sigma(\mathcal{N})$. We will often apply this completion procedure to the canonical filtration of a process X and call the resulting filtration the completed canonical filtration of X.

We say that a filtration $(\mathcal{F}_t)_{0\leq t\leq\infty}$ is right-continuous if

$$\mathcal{F}_{t+} = \mathcal{F}_t, \quad \text{for all } t \geq 0.$$

Let $(\mathcal{F}_t)_{0\leq t\leq\infty}$ be a filtration and let $\mathcal{N}$ be the collection of all $(\mathcal{F}_\infty, \mathbb{P})$-negligible sets ($A \in \mathcal{N}$ is there exists $A' \in \mathcal{F}_\infty$ such that $A \subset A'$ and $\mathbb{P}(A') = 0$). The filtration $(\mathcal{F}_t)_{0\leq t\leq\infty}$ is said to be complete if $\mathcal{N} \subset \mathcal{F}_0$.

If $(\mathcal{F}_t)_{0\leq t\leq\infty}$ is not complete, it can be completed by setting $\mathcal{F}'_t = \mathcal{F}_t \vee \sigma(\mathcal{N})$. We will often apply this completion procedure to the canonical filtration of a process X and call the resulting filtration the completed canonical filtration of X.

All results stated in Chap. 2, where we considered the canonical filtration of a Brownian motion *B*, remain valid if instead we deal with the completed canonical filtration. The point is that augmenting a σ-field with negligible sets does not alter independence properties.

Definition 3.2

A process $X = (X_t)_{t \geq 0}$ taking values in a measurable space $(E, \mathcal{E})$ is said to be measurable if the mapping

$$(\omega, t) \mapsto X_t(\omega)$$

is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.

All results stated in Chap. 2, where we considered the canonical filtration of a Brownian motion B, remain valid if instead we deal with the completed canonical filtration. The point is that augmenting a σ-field with negligible sets does not alter independence properties.

Definition 3.2

A process $X = (X_t)_{t \geq 0}$ taking values in a measurable space $(E, \mathcal{E})$ is said to be measurable if the mapping

$$(\omega, t) \mapsto X_t(\omega)$$

is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.

In the remaining part of this chapter, we fix a filtration $(\mathcal{F}_t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$, and the notions that will be introduced depend on the choice of this filtration.

Definition 3.3

A process $X = (X_t)_{t \geq 0}$ taking values in a measurable space $(E, \mathcal{E})$ is said to be adapted if, for every $t \geq 0$, X_t is $\mathcal{F}_t$-measurable.
$X = (X_t)_{t \geq 0}$ is said to be progressively measurable if, for every $t \geq 0$, the mapping

$$\Omega \times [0, t] \ni (\omega, s) \mapsto X_t(\omega)$$

is $\mathcal{F}_t \otimes \mathcal{B}([0, t])$-measurable.

Progressive measurability implies adaptedness and measurability.

In the remaining part of this chapter, we fix a filtration $(\mathcal{F}_t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$, and the notions that will be introduced depend on the choice of this filtration.

Definition 3.3

A process $X = (X_t)_{t \geq 0}$ taking values in a measurable space $(E, \mathcal{E})$ is said to be adapted if, for every $t \geq 0$, X_t is $\mathcal{F}_t$-measurable.
$X = (X_t)_{t \geq 0}$ is said to be progressively measurable if, for every $t \geq 0$, the mapping

$$\Omega \times [0, t] \ni (\omega, s) \mapsto X_t(\omega)$$

is $\mathcal{F}_t \otimes \mathcal{B}([0, t])$-measurable.

Progressive measurability implies adaptedness and measurability.

In the remaining part of this chapter, we fix a filtration $(\mathcal{F}_t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$, and the notions that will be introduced depend on the choice of this filtration.

Definition 3.3

A process $X = (X_t)_{t\geq 0}$ taking values in a measurable space $(E, \mathcal{E})$ is said to be adapted if, for every $t \geq 0$, X_t is $\mathcal{F}_t$-measurable.
$X = (X_t)_{t\geq 0}$ is said to be progressively measurable if, for every $t \geq 0$, the mapping

$$\Omega \times [0, t] \ni (\omega, s) \mapsto X_t(\omega)$$

is $\mathcal{F}_t \otimes \mathcal{B}([0, t])$-measurable.

Progressive measurability implies adaptedness and measurability.

Proposition 3.4

Let $X = (X_t)_{t \geq 0}$ be a process taking values in a metric space (E, d). Suppose that X is adapted and that the sample paths of X are right-continuous (i.e., for every $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is right-continuous. Then X is progressively measurable. The same conclusion holds if one replaces right-continuity by left-continuity.

Proof of Proposition 3.4

Fix $t > 0$. For every $n \geq 1$ and $s \in [0, t]$, define

$$X_s^n = X_{(kt)/n} \quad \text{if } s \in [\frac{(k-1)t}{n}, \frac{kt}{n}], k \in \{1, \ldots, n\}$$

and $X_t^n = X_t$. Then by right-continuity, for any $s \in [0, t]$ and any $\omega \in \Omega$,

$$X_s(\omega) = \lim_{n \to \infty} X_s^n(\omega).$$

Proposition 3.4

Let $X = (X_t)_{t \geq 0}$ be a process taking values in a metric space (E, d). Suppose that X is adapted and that the sample paths of X are right-continuous (i.e., for every $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is right-continuous. Then X is progressively measurable. The same conclusion holds if one replaces right-continuity by left-continuity.

Proof of Proposition 3.4

Fix $t > 0$. For every $n \geq 1$ and $s \in [0, t]$, define

$$X_s^n = X_{(kt)/n} \quad \text{if } s \in [\frac{(k-1)t}{n}, \frac{kt}{n}], k \in \{1, \ldots, n\}$$

and $X_t^n = X_t$. Then by right-continuity, for any $s \in [0, t]$ and any $\omega \in \Omega$,

$$X_s(\omega) = \lim_{n \to \infty} X_s^n(\omega).$$

Proof of Proposition 3.4

For any $a \in \mathcal{B}(E)$,

$$\begin{aligned}
&\{(\omega, s) \in \Omega \times [0, t] : X_s(\omega) \in A\} \\
&= (\{X_t \in A\} \times \{t\}) \cup \left(\cup_{k=1}^{n} \{X_{(kt)/n} \in A\} \times \left[\frac{(k-1)t}{n}, \frac{kt}{n} \right) \right) \\
&\in \mathcal{F}_t \otimes \mathcal{B}([0, t]).
\end{aligned}$$