Math 562 Fall 2020

Renming Song

University of Illinois at Urbana-Champaign

September 09, 2020

Outline

Outline

Proposition 2.16

Let $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ be a sequence of partitions of $[0, t]$ whose mesh tends to 0 (i.e., $\sup_{1 \le j \le p_n}(t_j^n - t_{j-1}^n) = 0$ as $n \to \infty$). Then

$$\lim_{n\to\infty} \sum_{j=1}^{p_n} (B_{t_j^n} - B_{t_{j-1}^n})^2 = t$$

in L^2.

Proof of Proposition 2.16

This is an immediate consequence of Proposition 1.14, writing $B_{t_j^n} - B_{t_{j-1}^n} = G((t_{j-1}^n, t_j^n])$, where G is the Gaussian white noise associated with B.

Proposition 2.16

Let $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ be a sequence of partitions of $[0, t]$ whose mesh tends to 0 (i.e., $\sup_{1 \le j \le p_n}(t_j^n - t_{j-1}^n) = 0$ as $n \to \infty$). Then

$$\lim_{n \to \infty} \sum_{j=1}^{p_n} (B_{t_j^n} - B_{t_{j-1}^n})^2 = t$$

in L^2.

Proof of Proposition 2.16

This is an immediate consequence of Proposition 1.14, writing $B_{t_j^n} - B_{t_{j-1}^n} = G((t_{j-1}^n, t_j^n])$, where G is the Gaussian white noise associated with B.

If $a < b$ and f a real-valued function defined on $[a, b]$, the function f is said to have infinite variation if the supremum of $\sum_{j=1}^{p} |f(t_j) - f(t_{j-1})|$, over all partitions $a = t_0 < t_1 < \cdots < t_p = b$, is infinite.

Corollary 2.17

Almost surely, the function $t \mapsto B_t$ is of infinite variation on any non-trivial interval.

If $a < b$ and f a real-valued function defined on $[a, b]$, the function f is said to have infinite variation if the supremum of $\sum_{j=1}^{p} |f(t_j) - f(t_{j-1})|$, over all partitions $a = t_0 < t_1 < \cdots < t_p = b$, is infinite.

Corollary 2.17

Almost surely, the function $t \mapsto B_t$ is of infinite variation on any non-trivial interval.

Proof of Corollary 2.17

From the simple Markov property, it suffices to consider the interval $[0, t]$ for some fixed $t > 0$. Using Proposition 2.16 and extracting a subsequence if necessary, we my assume that the convergence in Proposition 2.16 holds a.s. Now note that

$$\sum_{j=1}^{p_n}(B_{t_j^n} - B_{t_{j-1}^n})^2 \leq \left(\sup_{1\leq j\leq p_n} |B_{t_j^n} - B_{t_{j-1}^n}|\right)\sum_{j=1}^{p_n}|B_{t_j^n} - B_{t_{j-1}^n}|.$$

The lhs tends to t, the first factor on the rhs also goes to 0, so the 2nd factor on the rhs must go to ∞.

Thus it is impossible to define $\int_0^t f(s)dB_s$ as a pathwise Lebesgue-Stieltjes integral.

Proof of Corollary 2.17

From the simple Markov property, it suffices to consider the interval $[0, t]$ for some fixed $t > 0$. Using Proposition 2.16 and extracting a subsequence if necessary, we my assume that the convergence in Proposition 2.16 holds a.s. Now note that

$$\sum_{j=1}^{p_n} (B_{t_j^n} - B_{t_{j-1}^n})^2 \leq \left(\sup_{1 \leq j \leq p_n} |B_{t_j^n} - B_{t_{j-1}^n}| \right) \sum_{j=1}^{p_n} |B_{t_j^n} - B_{t_{j-1}^n}|.$$

The lhs tends to t, the first factor on the rhs also goes to 0, so the 2nd factor on the rhs must go to ∞.

Thus it is impossible to define $\int_0^t f(s)dB_s$ as a pathwise Lebesgue-Stieltjes integral.

Outline

We fix a Brownian motion $B = (B_t)_{t\geq 0}$. Recall that $\mathcal{F}_t = \sigma(B_s : s \in [0,t])$. We define $\mathcal{F}_\infty = \sigma(B_s : s \geq 0)$.

Definition 2.18

A random variable T with values in $[0,\infty]$ is said to be a stopping time if $\{T \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$.

Note that ∞ is an allowed value. If T is a stopping time, we also have, for every $t > 0$,

$$\{T < t\} = \cup_{q\in[0,t)\cap\mathbb{Q}}\{T \leq q\} \in \mathcal{F}_t.$$

We fix a Brownian motion $B = (B_t)_{t\geq 0}$. Recall that $\mathcal{F}_t = \sigma(B_s : s \in [0, t])$. We define $\mathcal{F}_\infty = \sigma(B_s : s \geq 0)$.

Definition 2.18

A random variable T with values in $[0, \infty]$ is said to be a stopping time if $\{T \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$.

Note that ∞ is an allowed value. If T is a stopping time, we also have, for every $t > 0$,

$$\{T < t\} = \cup_{q\in[0,t)\cap\mathbb{Q}}\{T \leq q\} \in \mathcal{F}_t.$$

We fix a Brownian motion $B = (B_t)_{t \geq 0}$. Recall that $\mathcal{F}_t = \sigma(B_s : s \in [0, t])$. We define $\mathcal{F}_\infty = \sigma(B_s : s \geq 0)$.

Definition 2.18

A random variable T with values in $[0, \infty]$ is said to be a stopping time if $\{T \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$.

Note that ∞ is an allowed value. If T is a stopping time, we also have, for every $t > 0$,

$$\{T < t\} = \cup_{q \in [0,t) \cap \mathbb{Q}} \{T \leq q\} \in \mathcal{F}_t.$$

Examples

(1) Positive constants are stopping times.

(2) For any $a \in \mathbb{R}$, $T_a = \inf\{t \geq 0 : B_t = a\}$ is a stopping time.

(3) If T is a stopping time, then for any $a \geq 0$, $T + a$ is also a stopping time.

(4) $T = \sup\{s \leq 1 : B_s = 0\}$ is not a stopping time.

Definition 2.19

Let T be a stopping time. We define

$$\mathcal{F}_T = \{A \in \mathcal{F}_\infty : A \cap \{T \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

$\mathcal{F}_T$ is a σ-field and it contains all the info up to time T.

Examples

(1) Positive constants are stopping times.

(2) For any $a \in \mathbb{R}$, $T_a = \inf\{t \geq 0 : B_t = a\}$ is a stopping time.

(3) If T is a stopping time, then for any $a \geq 0$, $T + a$ is also a stopping time.

(4) $T = \sup\{s \leq 1 : B_s = 0\}$ is not a stopping time.

Definition 2.19

Let T be a stopping time. We define

$$\mathcal{F}_T = \{A \in \mathcal{F}_\infty : A \cap \{T \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

$\mathcal{F}_T$ is a σ-field and it contains all the info up to time T.

If T is a stopping time, then the random variable $1_{\{T<\infty\}}B_T$ is $\mathcal{F}_T$-measurable. Indeed, note that

$$1_{\{T<\infty\}}B_T = \lim_{n\to\infty}\sum_{j=0}^{\infty} 1_{\{j2^{-n}\le T<(j+1)2^{-n}\}}B_{j2^{-n}}$$

$$= \lim_{n\to\infty}\sum_{j=0}^{\infty} 1_{\{T<(j+1)2^{-n}\}}1_{\{j2^{-n}\le T\}}B_{j2^{-n}}.$$

We will show that, for any $s\ge 0$, $B_s 1_{\{s\le T\}}\in\mathcal{F}_T$. Once we have this, we get $1_{\{T<\infty\}}B_T$ is $\mathcal{F}_T$-measurable.

If $A\in\mathcal{B}(\mathbb{R})$ and $0\notin A$, then

$$\{B_s 1_{\{s\le T\}}\in A\}\cap\{T\le t\} = \begin{cases}\emptyset, & t<s\\ \{B_s\in A\}\cap\{s\le T\le t\}, & t\ge s\end{cases}$$

is in $\mathcal{F}_t$. If $A\in\mathcal{B}(\mathbb{R})$ and $0\in A$, we consider the complement.

If T is a stopping time, then the random variable $1_{\{T<\infty\}} B_T$ is $\mathcal{F}_T$-measurable. Indeed, note that

$$
\begin{aligned}
1_{\{T<\infty\}} B_T &= \lim_{n\to\infty} \sum_{j=0}^{\infty} 1_{\{j2^{-n} \le T < (j+1)2^{-n}\}} B_{j2^{-n}} \\
&= \lim_{n\to\infty} \sum_{j=0}^{\infty} 1_{\{T<(j+1)2^{-n}\}} 1_{\{j2^{-n} \le T\}} B_{j2^{-n}}.
\end{aligned}
$$

We will show that, for any $s \ge 0$, $B_s 1_{\{s \le T\}} \in \mathcal{F}_T$. Once we have this, we get $1_{\{T<\infty\}} B_T$ is $\mathcal{F}_T$-measurable.

If $A \in \mathcal{B}(\mathbb{R})$ and $0 \notin A$, then

$$
\{B_s 1_{\{s\le T\}} \in A\} \cap \{T \le t\} = \begin{cases} \emptyset, & t < s \\ \{B_s \in A\} \cap \{s \le T \le t\}, & t \ge s \end{cases}
$$

is in $\mathcal{F}_t$. If $A \in \mathcal{B}(\mathbb{R})$ and $0 \in A$, we consider the complement.

Theorem 2.20 (Strong Markov Property)

Let T be a stopping time. We assume that $\mathbb{P}(T < \infty) > 0$ and define, for every $t \geq 0$,

$$B_t^{(T)} = 1_{\{T<\infty\}}(B_{T+t} - B_T).$$

Then under the probability measure $\mathbb{P}(\cdot | T < \infty)$, the process $(B_t^{(T)})_{t\geq 0}$ is a Brownian motion independent of $\mathcal{F}_T$.

Proof of Theorem 2.20

We first consider the case where $\mathbb{P}(T < \infty) = 1$. Fix $A \in \mathcal{F}_T$ and $0 \leq t_1 < \cdots < t_p$, and let F be a bounded continuous function from $\mathbb{R}^p$ to $\mathbb{R}$. We will first show that

Theorem 2.20 (Strong Markov Property)

Let T be a stopping time. We assume that $\mathbb{P}(T < \infty) > 0$ and define, for every $t \geq 0$,

$$B_t^{(T)} = 1_{\{T<\infty\}}(B_{T+t} - B_T).$$

Then under the probability measure $\mathbb{P}(\cdot | T < \infty)$, the process $(B_t^{(T)})_{t\geq 0}$ is a Brownian motion independent of $\mathcal{F}_T$.

Proof of Theorem 2.20

We first consider the case where $\mathbb{P}(T < \infty) = 1$. Fix $A \in \mathcal{F}_T$ and $0 \leq t_1 < \cdots < t_p$, and let F be a bounded continuous function from $\mathbb{R}^p$ to $\mathbb{R}$. We will first show that

Proof of Theorem 2.20 (cont)

$$\mathbb{E}[1_A F(B_{t_1}^{(T)}, \ldots, B_{t_p}^{(T)})] = \mathbb{P}(A)\mathbb{E}[F(B_{t_1}, \ldots, B_{t_p})]. \tag{1}$$

Once we prove this, the conclusions of the theorem in this case will follow. In fact, taking $A = \Omega$ yields that $(B_t^{(T)})_{t\geq 0}$ is a Brownian motion. Independence of $(B_t^{(T)})_{t\geq 0}$ and $\mathcal{F}_T$ is a consequence of (1).
Now we prove (1). For every $t \geq 0$, we define

$$[t]_n = \min\{k2^{-n} : k \in \mathbb{Z}_+ \text{ and } k2^{-n} \geq t\}$$

and $[\infty]_n = \infty$. Note that

$$F(B_{t_1}^{(T)}, \ldots, B_{t_p}^{(T)}) = \lim_{n\to\infty} F(B_{t_1}^{([T]_n)}, \ldots, B_{t_p}^{([T]_n)}).$$

Hence by the dominated convergence theorem

Proof of Theorem 2.20 (cont)

$$\mathbb{E}[1_A F(B_{t_1}^{(T)}, \ldots, B_{t_p}^{(T)})] = \lim_{n\to\infty} \mathbb{E}[1_A F(B_{t_1}^{([T]_n)}, \ldots, B_{t_p}^{([T]_n)})]$$

$$= \lim_{n\to\infty} \sum_{k=0}^{\infty} \mathbb{E}[1_A 1_{\{(k-1)2^{-n} < T \le k2^{-n}\}} F(B_{k2^{-n}+t_1} - B_{k2^{-n}}, \ldots, B_{k2^{-n}+t_p} - B_{k2^{-n}})]$$

$$\stackrel{MP}{=} \lim_{n\to\infty} \sum_{k=0}^{\infty} \mathbb{P}(A \cap \{(k-1)2^{-n} < T \le k2^{-n}\})\mathbb{E}[F(B_{t_1}, \ldots, B_{t_p})]$$

$$= \mathbb{P}(A)\mathbb{E}[F(B_{t_1}, \ldots, B_{t_p})].$$

Thus (1) is valid.

Proof of Theorem 2.20 (cont)

Finally, when $\mathbb{P}(T < \infty) \in (0, 1)$, the same argument gives

$$\mathbb{E}[1_A 1_{\{T<\infty\}} F(B_{t_1}^{(T)}, \ldots, B_{t_p}^{(T)})] = \mathbb{P}(A \cap \{T < \infty\})\mathbb{E}[F(B_{t_1}, \ldots, B_{t_p})].$$

The desired results also follow easily in this case.

An important application of the strong Markov property is the"reflection principle" that leads to the following theorem.

Proof of Theorem 2.20 (cont)

Finally, when $\mathbb{P}(T < \infty) \in (0, 1)$, the same argument gives

$$\mathbb{E}[1_A 1_{\{T<\infty\}} F(B_{t_1}^{(T)}, \ldots, B_{t_p}^{(T)})] = \mathbb{P}(A \cap \{T < \infty\})\mathbb{E}[F(B_{t_1}, \ldots, B_{t_p})].$$

The desired results also follow easily in this case.

An important application of the strong Markov property is the"reflection principle" that leads to the following theorem.

Theorem 2.21

For $t > 0$, set $S_t = \sup_{s \leq t} B_s$. Then, if $a \geq 0$ and $b \in (-\infty, a]$, we have

$$\mathbb{P}(S_t \geq a, B_t \leq b) = \mathbb{P}(B_t \geq 2a - b).$$

In particular, S_t has the same distribution as $|B_t|$.

Proof of Theorem 2.21

We apply the strong Markov property at the stopping time

$$T_a = \inf\{t \geq 0 : B_t = a\}.$$

Note that $\mathbb{P}(T_a < \infty) = 1$.

Theorem 2.21

For $t > 0$, set $S_t = \sup_{s \leq t} B_s$. Then, if $a \geq 0$ and $b \in (-\infty, a]$, we have

$$\mathbb{P}(S_t \geq a, B_t \leq b) = \mathbb{P}(B_t \geq 2a - b).$$

In particular, S_t has the same distribution as $|B_t|$.

Proof of Theorem 2.21

We apply the strong Markov property at the stopping time

$$T_a = \inf\{t \geq 0 : B_t = a\}.$$

Note that $\mathbb{P}(T_a < \infty) = 1$.

Proof of Theorem 2.21 (cont)

Using the notation of Theorem 2.20, we have

$$\mathbb{P}(S_t \geq a, B_t \leq b) = \mathbb{P}(T_a \leq t, B_t \leq b) = \mathbb{P}(T_a \leq t, B^{(T_a)}_{t-T_a} \leq b - a)$$

since $B^{(T_a)}_{t-T_a} = B_t - B_{T_a} = B_t - a$ on the event $\{T_a \leq t\}$. Write $B' = B^{(T_a)}$, then by the strong Markov property, B' is a Brownian motion independent of $\mathcal{F}_{T_a}$ and hence independent of T_a. Since B' has the same law as $-B'$, the pair (T_a, B') also has the same law as $(T_a, -B')$. Let

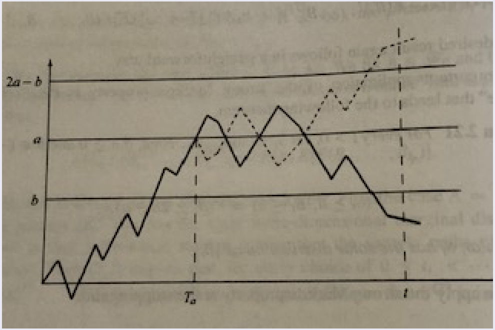
$$H = \{(s, w) \in \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}) : s \leq t, w(t - s) \leq b - a\}.$$

Then $\mathbb{P}(T_a \leq t, B^{(T_a)}_{t-T_a} \leq b - a)$ is equal to

Proof of Theorem 2.21 (cont)

$$\begin{aligned}
\mathbb{P}((T_a, B') \in H)) &= \mathbb{P}((T_a, -B') \in H) \\
&= \mathbb{P}(T_a \le t, -B^{(T_a)}_{t-T_a} \le b-a) = \mathbb{P}(T_a \le t, B_t \ge 2a-b) \\
&= \mathbb{P}(B_t \ge 2a-b)
\end{aligned}$$

since $\{B_t \ge 2a - b\} \subset \{T_a \le t\}$. This gives the first assertion.



Proof of Theorem 2.21 (cont)

For the last assertion of the theorem, we observe that

$$\begin{aligned}\mathbb{P}(S_t \geq a) &= \mathbb{P}(S_t \geq a, B_t \geq a) + \mathbb{P}(S_t \geq a, B_t \leq a) \\ &= 2\mathbb{P}(B_t \geq a) = \mathbb{P}(|B_t| \geq a)\end{aligned}$$

and the desired result follows.

It follows from the previous theorem that the law (S_t, B_t) has density

$$g(a,b) = \frac{2(2a-b)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2a-b)^2}{2t}\right) 1_{\{a>0, b<a\}}.$$

Proof of Theorem 2.21 (cont)

For the last assertion of the theorem, we observe that

$$\begin{aligned}\mathbb{P}(S_t \geq a) &= \mathbb{P}(S_t \geq a, B_t \geq a) + \mathbb{P}(S_t \geq a, B_t \leq a)\\ &= 2\mathbb{P}(B_t \geq a) = \mathbb{P}(|B_t| \geq a)\end{aligned}$$

and the desired result follows.

It follows from the previous theorem that the law (S_t, B_t) has density

$$g(a,b) = \frac{2(2a-b)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2a-b)^2}{2t}\right) 1_{\{a>0, b<a\}}.$$

Corollary 2.22

For any $a > 0$, T_a has the same distribution as $\frac{a^2}{B_1^2}$ and has density

$$f(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right) 1_{\{a>0\}}.$$

Proof of Corollary 2.22

For any $t \geq 0$,

$$\mathbb{P}(T_a \leq t) = \mathbb{P}(S_t \geq a) = \mathbb{P}(|B_t| \geq a) = P(B_t^2 \geq a^2)$$
$$= \mathbb{P}(tB_1^2 \geq a^2) = \mathbb{P}(\frac{a^2}{B_1^2} \leq t).$$

A straight foward calculation gives the density of $\frac{a^2}{B_1^2}$.

Corollary 2.22

For any $a > 0$, T_a has the same distribution as $\frac{a^2}{B_1^2}$ and has density

$$f(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right) 1_{\{a>0\}}.$$

Proof of Corollary 2.22

For any $t \geq 0$,

$$\mathbb{P}(T_a \leq t) = \mathbb{P}(S_t \geq a) = \mathbb{P}(|B_t| \geq a) = P(B_t^2 \geq a^2)$$
$$= \mathbb{P}(tB_1^2 \geq a^2) = \mathbb{P}(\frac{a^2}{B_1^2} \leq t).$$

A straight foward calculation gives the density of $\frac{a^2}{B_1^2}$.