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1. General Info

2. 2.2 Continuity of Sample Paths

3. 2.3 Properties of Brownian Sample Paths.
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Proposition 2.5, which was stated for pre-Brownian motion, remains true for Brownian motions.

**Proposition 2.5**

Let \((B_t)_{t \geq 0}\) be Brownian motion. Then

(i) \((-B_t)_{t \geq 0}\) is a Brownian motion;

(ii) for every \(\lambda > 0\), \((\frac{1}{\lambda} B_{\lambda^2 t})_{t \geq 0}\) is a Brownian motion;

(iii) for every \(s \geq 0\), \((B_{s+t} - B_s)_{t \geq 0}\) is a Brownian motion and is independent of \(\sigma(B_r : r \leq s)\).

Property (iii) above is also known as the (simple) Markov property.
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Property (iii) above is also known as the (simple) Markov property.
Let $C(\mathbb{R}_+, \mathbb{R})$ be the space of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$. We equip $C(\mathbb{R}_+, \mathbb{R})$ with the $\sigma$-field $\mathcal{C}$ defined as the smallest $\sigma$-field on $C(\mathbb{R}_+, \mathbb{R})$ for which the coordinate mappings $w \mapsto w(t)$ are measurable for every $t \geq 0$.

$\mathcal{C}$ is generated by the cylinder sets:

$$\{ w \in C(\mathbb{R}_+, \mathbb{R}) : w(t_1) \in A_1, \ldots, w(t_n) \in A_n \},$$

where $n \geq 1$, $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ and $A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})$. 
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where $n \geq 1$, $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ and $A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})$. 
Equipped with the metric

\[ d(w_1, w_2) = \sum_{n=1}^{\infty} 2^{-n} \max_{0 \leq t \leq n} (|w_1(t) - w_2(t)| \wedge 1), \]

\( C(\mathbb{R}_+, \mathbb{R}) \) is a complete, separable metric space. This topology is the topology of uniform convergence on compact subsets of \( \mathbb{R}_+ \).

\( \mathcal{C} \) is equal to the Borel \( \sigma \)-field on the metric space \( (C(\mathbb{R}_+, \mathbb{R}), d) \).
(Think about a proof!)
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(Think about a proof!)
Given a Brownian motion \( B = (B_t)_{t \geq 0} \), we can consider it as a mapping
\[
\Omega \ni \omega \mapsto (t \mapsto B_t(\omega)) \in C(\mathbb{R}_+, \mathbb{R}).
\]
This map is measurable (the inverse image of any cylinder sets are in \( \mathcal{F} \).

**Definition**

The Wiener measure \( W(dw) \) is the image of the probability measure \( \mathbb{P}(d\omega) \) under this mapping. For any \( A \in \mathcal{C} \),

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W(A) = \mathbb{P}(B. \in A).
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When $A = \{ w \in C(\mathbb{R}_+, \mathbb{R}) : w(t_0) \in A_0, w(t_1) \in A_1, \ldots, w(t_n) \in A_n \}$ with $n \geq 1$, $0 = t_0 < t_1 < t_2 < \cdots < t_n < \infty$ and $A_0, A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})$, Corollary 2.4 gives

$$W(A) = \mathbb{P}(B_{t_0} \in A_0, B_{t_1} \in A_1, \ldots, B_{t_n} \in A_n)$$

$$= 1_{A_0}(0) \int_{A_1 \times \cdots \times A_n} \frac{dx_1 \cdots dx_n}{(2\pi)^{n/2} \sqrt{t_1(t_2 - t_1) \cdots (t_n - t_{n-1})}} e^{-\sum_{i=1}^{n} \frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}}$$

where $x_0 = 0$.

Suppose $B' = (B'_t)_{t \geq 0}$ is another Brownian motion. Then for any $A \in \mathcal{C}$,

$$\mathbb{P}'(B' \in A) = W(A) = \mathbb{P}(B_t \in A).$$
When \( A = \{ w \in C(\mathbb{R}^+, \mathbb{R}) : w(t_0) \in A_0, w(t_1) \in A_1, \ldots, w(t_n) \in A_n \} \)
with \( n \geq 1, 0 = t_0 < t_1 < t_2 < \cdots < t_n < \infty \) and \( A_0, A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R}) \),
Corollary 2.4 gives

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\]
where \( x_0 = 0 \).

Suppose \( B' = (B'_t)_{t \geq 0} \) is another Brownian motion. Then for any \( A \in \mathcal{C} \),

\[
\mathbb{P}'(B' \in A) = W(A) = \mathbb{P}(B \in A).
\]
Thus the probability that a given property (corresponding to a measurable subset $A$ of $C(\mathbb{R}_+, \mathbb{R})$) holds is the same for the sample paths of $B$ and for the sample paths of $B'$.

Consider now the special choice of a probability space,

$$\Omega = C(\mathbb{R}_+, \mathbb{R}), \quad \mathcal{F} = C, \quad \mathbb{P} = \mathcal{W}.$$ 

Then on this probability space, the canonical process (or coordinate process)

$$X_t(w) = w(t)$$

is a Brownian motion. This is called a canonical Brownian motion.
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3. 2.3 Properties of Brownian Sample Paths.
In this section, we fix a Brownian motion \( B = (B_t)_{t \geq 0} \). For \( t \geq 0 \), we define

\[ \mathcal{F}_t = \sigma(B_s : s \leq t). \]

Note that \( s \leq t \) implies \( \mathcal{F}_s \subset \mathcal{F}_t \). We also define

\[ \mathcal{F}_{0+} = \cap_{s > 0} \mathcal{F}_s. \]

**Theorem 2.13 (Blumenthal’s 0-1 law)**

\( \mathcal{F}_{0+} \) is trivial, that is, \( \mathbb{P}(A) \in \{0, 1\} \) for every \( A \in \mathcal{F}_{0+} \).
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**Theorem 2.13 (Blumenthal’s 0-1 law)**

$\mathcal{F}_{0+}$ is trivial, that is, $\mathbb{P}(A) \in \{0, 1\}$ for every $A \in \mathcal{F}_{0+}$.
Proof of Theorem 2.13

Let $0 < t_1 < \cdots < t_k$ and let $g : \mathbb{R}^k \to \mathbb{R}$ be a bounded continuous function. Fix $A \in \mathcal{F}_{0+}$. By continuity and the bounded convergence theorem,

$$
\mathbb{E}[1_A g(B_{t_1}, \ldots, B_{t_k})] = \lim_{\epsilon \downarrow 0} \mathbb{E}[1_A g(B_{t_1} - B_{\epsilon}, \ldots, B_{t_k} - B_{\epsilon})].
$$

If $\epsilon \in (0, t_1)$, then random variables $B_{t_1} - B_{\epsilon}, \ldots, B_{t_k} - B_{\epsilon}$ are independent of $\mathcal{F}_\epsilon$, and thus also of $\mathcal{F}_{0+}$. It follows that

$$
\mathbb{E}[1_A g(B_{t_1}, \ldots, B_{t_k})] = P(A) \lim_{\epsilon \downarrow 0} \mathbb{E}[g(B_{t_1} - B_{\epsilon}, \ldots, B_{t_k} - B_{\epsilon})]
$$

$$
= P(A) \mathbb{E}[g(B_{t_1}, \ldots, B_{t_k})].
$$

Thus $\mathcal{F}_{0+}$ is independent of $\sigma(B_{t_1}, \ldots, B_{t_k})$. Since this holds for finite collection $\{t_1, \ldots, t_k\}$ of strictly positive numbers, $\mathcal{F}_{0+}$ is independent of $\sigma(B_t : t > 0)$. However, $\sigma(B_t : t > 0) = \sigma(B_t : t \geq 0)$. Thus $\mathcal{F}_{0+}$ is independent of itself.
Proposition 2.14

(i) We have, a.s., for every $\varepsilon > 0$,

$$\sup_{0 \leq s \leq \varepsilon} B_s > 0, \quad \inf_{0 \leq s \leq \varepsilon} B_s < 0.$$ 

(ii) For every $a \in \mathbb{R}$, let $T_a = \inf\{t \geq 0 : B_t = a\}$. Then

$$a.s., \text{ for any } a \in \mathbb{R}, \quad T_a < \infty.$$ 

Consequently, we have a.s.

$$\limsup_{t \to \infty} B_t = \infty, \quad \liminf_{t \to \infty} B_t = -\infty.$$
Proof of Proposition 2.14

(i) Let $\epsilon_k > 0$ be a sequence strictly decreasing to 0, and let

$$A = \cap_p \left\{ \sup_{0 \leq s \leq \epsilon_p} B_s > 0 \right\}.$$ 

Obviously $A \in \mathcal{F}_{0+}$. On the other hand,

$$\mathbb{P}(A) = \lim_{p \to \infty} \mathbb{P}(\sup_{0 \leq s \leq \epsilon_p} B_s > 0)$$

and

$$\mathbb{P}(\sup_{0 \leq s \leq \epsilon_p} B_s > 0) \geq \mathbb{P}(B_{\epsilon_p} > 0) = \frac{1}{2},$$

which implies $\mathbb{P}(A) \geq \frac{1}{2}$. By Theorem 2.13 we have $\mathbb{P}(A) = 1$, hence

$$\text{a.s. for every } \epsilon > 0, \sup_{0 \leq s \leq \epsilon} B_s > 0.$$ 

The assertion about $\inf_{0 \leq s \leq \epsilon} B_s$ is obtained by replacing $B$ by $-B$. 
Proof of Proposition 2.14 (cont)

(ii) We know that

\[ 1 = \mathbb{P}( \sup_{0 \leq s \leq 1} B_s > 0) = \lim_{\delta \downarrow 0} \mathbb{P}( \sup_{0 \leq s \leq 1} B_s > \delta). \]

By the scale invariance property \((B^\lambda_t = \frac{1}{\lambda} B_{\lambda^2 t})\) with \(\lambda = \delta\), we see that

\[ \mathbb{P}( \sup_{0 \leq s \leq 1} B_s > \delta) = \mathbb{P}( \sup_{0 \leq s \leq 1/\delta^2} B_{\delta}^\delta > 1) = \mathbb{P}( \sup_{0 \leq s \leq 1/\delta^2} B_s > 1). \]

Letting \(\delta \downarrow 0\), we get

\[ \mathbb{P}(\sup_{s \geq 0} B_s > 1) = \lim_{\delta \downarrow 0} \mathbb{P}( \sup_{0 \leq s \leq 1/\delta^2} B_{\delta}^\delta > 1) = 1. \]

Then another scaling property shows that, for any \(M > 0\),

\[ \mathbb{P}(\sup_{s \geq 0} B_s > M) = 1. \]
Corollary 2.15
Almost surely, the function $t \mapsto B_t$ is not monotone on any nontrivial interval.

Proof of Corollary 2.15
By Proposition 2.14(i) and the simple Markov property, we get that a.s. for any $q \in \mathbb{Q}_+$, for every $\epsilon > 0,$

$$\sup_{q \leq t \leq q + \epsilon} B_t > B_q, \quad \inf_{q \leq t \leq q + \epsilon} B_t < B_q.$$  

The desired property follows.
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By Proposition 2.14(i) and the simple Markov property, we get that a.s. for any $q \in \mathbb{Q}_+$, for every $\epsilon > 0$,

$$\sup_{q \leq t \leq q + \epsilon} B_t > B_q, \quad \inf_{q \leq t \leq q + \epsilon} B_t < B_q.$$  

The desired property follows.
**Proposition 2.16**

Let $0 = t_0^n < t_1^n < \cdots < t^n_{p_n} = t$ be a sequence of partitions of $[0, t]$ whose mesh tends to 0 (i.e., $\sup_{1 \leq j \leq p_n} (t^n_j - t^n_{j-1}) = 0$ as $n \to \infty$). Then

$$\lim_{n \to \infty} \sum_{j=1}^{p_n} (B^n_{t_j} - B^n_{t_{j-1}})^2 = t$$

in $L^2$.

**Proof of Proposition 2.16**

This is an immediate consequence of Proposition 1.14, writing $B^n_{t_j} - B^n_{t_{j-1}} = G((t^n_{j-1}, t^n_j))$, where $G$ is the Gaussian white noise associated with $B$. 
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This is an immediate consequence of Proposition 1.14, writing $B_{t_j^n} - B_{t_{j-1}^n} = G((t_{j-1}^n, t_j^n))$, where $G$ is the Gaussian white noise associated with $B$. 
If $a < b$ and $f$ a real-valued function defined on $[a, b]$, the function $f$ is said to have infinite variation if the supremum of $\sum_{j=1}^{p} |f(t_j) - f(t_{j-1})|$, over all partitions $a = t_0 < t_1 < \cdots < t_p = b$, is infinite.

**Corollary 2.17**

Almost surely, the function $t \mapsto B_t$ is of infinite variation on any non-trivial interval.
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**Corollary 2.17**

Almost surely, the function $t \mapsto B_t$ is of infinite variation on any non-trivial interval.
Proof of Corollary 2.17

From the simple Markov property, it suffices to consider the interval 
\([0, t]\) for some fixed \(t > 0\). Using Proposition 2.16 and extracting a 
subsequence if necessary, we may assume that the convergence in 
Proposition 2.16 holds a.s. Now note that

\[
\sum_{j=1}^{p_n} (B_{t_j^n} - B_{t_{j-1}^n})^2 \leq \left( \sup_{1 \leq j \leq p_n} |B_{t_j^n} - B_{t_{j-1}^n}| \right) \sum_{j=1}^{p_n} |B_{t_j^n} - B_{t_{j-1}^n}|.
\]

The lhs tends to 0, the first factor on the rhs also goes to 0, so the 
2nd factor on the rhs must go to \(\infty\).

Thus it is impossible to define \(\int_0^t f(s) dB_s\) as a pathwise 
Lebesgue-Stieltjes integral.
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