Math 562 Fall 2020

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Outline

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1. General Info
2. 2.2 Continuity of Sample Paths
3. 2.3 Properties of Brownian Sample Paths.

HW1 is posted on my homepage. I also setup HW1 in the course Moodle page. You need to submit your HW1 via Moodle. The due date for HW1 is 09/08 at noon.

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3. 2.3 Properties of Brownian Sample Paths.

Proposition 2.5, which was stated for pre-Brownian motion, remains true for Brownian motions.

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Let $(B_t)_{t\geq 0}$ be Brownian motion. Then

(i) $(-B_t)_{t\geq 0}$ is a Brownian motion;

(ii) for every $\lambda > 0$, $(\frac{1}{\lambda} B_{\lambda^2 t})_{t\geq 0}$ is a Brownian motion;

(iii) for every $s \geq 0$, $(B_{s+t} - B_s)_{t\geq 0}$ is a Brownian motion and is independent of $\sigma(B_r : r \leq s)$.

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Let $C(\mathbb{R}_+, \mathbb{R})$ be the space of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$. We equip $C(\mathbb{R}_+, \mathbb{R})$ with the σ-field $\mathcal{C}$ defined as the smallest σ-field on $C(\mathbb{R}_+, \mathbb{R})$ for which the coordinate mappings $w \mapsto w(t)$ are measurable for every $t \geq 0$.

$\mathcal{C}$ is generated by the cylinder sets:

$$\{w \in C(\mathbb{R}_+, \mathbb{R}) : w(t_1) \in A_1, \dots, w(t_n) \in A_n\},$$

where $n \geq 1$, $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ and $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$.

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Equipped with the metric

$$d(w_1, w_2) = \sum_{n=1}^{\infty} 2^{-n} \max_{0 \le t \le n} (|w_1(t) - w_2(t)| \wedge 1),$$

$C(\mathbb{R}_+, \mathbb{R})$ is a complete, separable metric space. This topology is the topology of uniform convergence on compact subsets of $\mathbb{R}_+$.

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Given a Brownian motion $B = (B_t)_{t \geq 0}$, we can consider it as a mapping

$$\Omega \ni \omega \mapsto (t \mapsto B_t(\omega)) \in C(\mathbb{R}_+, \mathbb{R}).$$

This map is measurable (the inverse image of any any cylinder sets are in $\mathcal{F}$.

Definition

The Wiener measure $W(dw)$ is the image of the probability measure $\mathbb{P}(d\omega)$ under this mapping. For any $A \in \mathcal{C}$,

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When $A = \{w \in C(\mathbb{R}_+, \mathbb{R}) : w(t_0) \in A_0, w(t_1) \in A_1, \ldots, w(t_n) \in A_n\}$ with $n \geq 1$, $0 = t_0 < t_1 < t_2 < \cdots < t_n < \infty$ and $A_0, A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})$, Corollary 2.4 gives

$$W(A) = \mathbb{P}(B_{t_0} \in A_0, B_{t_1} \in A_1, \ldots, B_{t_n} \in A_n)$$
$$= 1_{A_0}(0) \int_{A_1 \times \cdots \times A_n} \frac{dx_1 \cdots dx_n}{(2\pi)^{n/2}\sqrt{t_1(t_2 - t_1) \cdots (t_n - t_{n-1})}} e^{-\sum_{i=1}^n \frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}}$$

where $x_0 = 0$.

Suppose $B' = (B'_t)_{t \geq 0}$ is another Brownian motion. Then for any $A \in \mathcal{C}$,

$$\mathbb{P}'(B'_\cdot \in A) = W(A) = \mathbb{P}(B_\cdot \in A).$$

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Thus the probability that a given property (corresponding to a measurable subset A of $C(\mathbb{R}_+, \mathbb{R})$) holds is the same for the sample paths of B and for the sample paths of B'.

Consider now the special choice of a probability space,

$$\Omega = C(\mathbb{R}_+, \mathbb{R}), \quad \mathcal{F} = \mathcal{C}, \quad \mathbb{P} = W.$$

Then on this probability space, the canonical process (or coordinate process)

$$X_t(w) = w(t)$$

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2. 2.2 Continuity of Sample Paths
3. **2.3 Properties of Brownian Sample Paths.**

In this section, we fix a Brownian motion $B = (B_t)_{t \geq 0}$. For $t \geq 0$, we define

$$\mathcal{F}_t = \sigma(B_s : s \leq t).$$

Note that $s \leq t$ implies $\mathcal{F}_s \subset \mathcal{F}_t$. We also define

$$\mathcal{F}_{0+} = \cap_{s>0} \mathcal{F}_s.$$

Theorem 2.13 (Blumenthal's 0-1 law)

$\mathcal{F}_{0+}$ is trivial, that is, $\mathbb{P}(A) \in \{0, 1\}$ for every $A \in \mathcal{F}_{0+}$.

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Theorem 2.13 (Blumenthal's 0-1 law)

$\mathcal{F}_{0+}$ is trivial, that is, $\mathbb{P}(A) \in \{0, 1\}$ for every $A \in \mathcal{F}_{0+}$.

Proof of Theorem 2.13

Let $0 < t_1 < \cdots < t_k$ and let $g : \mathbb{R}^k \to \mathbb{R}$ be a bounded continuous function. Fix $A \in \mathcal{F}_{0+}$. By continuity and the bounded convergence theorem,

$$\mathbb{E}[1_A g(B_{t_1}, \ldots, B_{t_k})] = \lim_{\epsilon \downarrow 0} \mathbb{E}[1_A g(B_{t_1} - B_\epsilon, \ldots, B_{t_k} - B_\epsilon)].$$

If $\epsilon \in (0, t_1)$, then random variables $B_{t_1} - B_\epsilon, \ldots B_{t_k} - B_\epsilon$ are independent of $\mathcal{F}_\epsilon$, and thus also of $\mathcal{F}_{0+}$. It follows that

$$\begin{aligned}\mathbb{E}[1_A g(B_{t_1}, \ldots, B_{t_k})] =& P(A) \lim_{\epsilon \downarrow 0} \mathbb{E}[g(B_{t_1} - B_\epsilon, \ldots, B_{t_k} - B_\epsilon)] \\ =& P(A)\mathbb{E}[g(B_{t_1}, \ldots, B_{t_k})].\end{aligned}$$

Thus $\mathcal{F}_{0+}$ is independent of $\sigma(B_{t_1}, \ldots, B_{t_k})$. Since this holds for finite collection $\{t_1, \ldots, t_k\}$ of strictly positive numbers, $\mathcal{F}_{0+}$ is independent of $\sigma(B_t : t > 0)$. However, $\sigma(B_t : t > 0) = \sigma(B_t : t \geq 0)$. Thus $\mathcal{F}_{0+}$ is independent of itself.

Proposition 2.14

(i) We have, a.s., for every $\epsilon > 0$,

$$\sup_{0\le s\le\epsilon} B_s > 0, \qquad \inf_{0\le s\le\epsilon} B_s < 0.$$

(ii) For every $a \in \mathbb{R}$, let $T_a = \inf\{t \ge 0 : B_t = a\}$. Then

$$a.s., \text{ for any } a \in \mathbb{R}, \quad T_a < \infty.$$

Consequently, we have a.s.

$$\limsup_{t\to\infty} B_t = \infty, \qquad \liminf_{t\to\infty} B_t = -\infty.$$

Proof of Proposition 2.14

(i) Let $\epsilon_k > 0$ be a sequence strictly decreasing to 0, and let

$$A = \cap_p \left\{ \sup_{0 \leq s \leq \epsilon_p} B_s > 0 \right\}.$$

Obviously $A \in \mathcal{F}_{0+}$. On the other hand,

$$\mathbb{P}(A) = \lim_{p \to \infty} \mathbb{P}(\sup_{0 \leq s \leq \epsilon_p} B_s > 0)$$

and

$$\mathbb{P}(\sup_{0 \leq s \leq \epsilon_p} B_s > 0) \geq \mathbb{P}(B_{\epsilon_p} > 0) = \frac{1}{2},$$

which implies $\mathbb{P}(A) \geq \frac{1}{2}$. By Theorem 2.13 we have $\mathbb{P}(A) = 1$, hence

$$a.s. \text{ for every } \epsilon > 0, \sup_{0 \leq s \leq \epsilon} B_s > 0.$$

The assertion about $\inf_{0 \leq s \leq \epsilon} B_s$ is obtained by replacing B by $-B$.

Proof of Proposition 2.14 (cont)

(ii) We know that

$$1 = \mathbb{P}(\sup_{0\le s\le 1} B_s > 0) = \lim_{\delta\downarrow 0} \uparrow \mathbb{P}(\sup_{0\le s\le 1} B_s > \delta).$$

By the scale invariance property ($B_t^\lambda = \frac{1}{\lambda} B_{\lambda^2 t}$) with $\lambda = \delta$, we see that

$$\mathbb{P}(\sup_{0\le s\le 1} B_s > \delta) = \mathbb{P}(\sup_{0\le s\le 1/\delta^2} B_s^\delta > 1) = \mathbb{P}(\sup_{0\le s\le 1/\delta^2} B_s > 1).$$

Letting $\delta \downarrow 0$, we get

$$\mathbb{P}(\sup_{s\ge 0} B_s > 1) = \lim_{\delta\downarrow 0} \uparrow \mathbb{P}(\sup_{0\le s\le 1/\delta^2} B_s^\delta > 1) = 1.$$

Then another scaling property shows that, for any $M > 0$,

$$\mathbb{P}(\sup_{s\ge 0} B_s > M) = 1.$$

Corollary 2.15

Almost surely, the function $t \mapsto B_t$ is not monotone on any nontrivial interval.

Proof of Corollary 2.15

By Proposition 2.14(i) and the simple Markov property, we get that a.s. for any $q \in \mathbb{Q}_+$, for every $\epsilon > 0$,

$$\sup_{q \le t \le q+\epsilon} B_t > B_q, \qquad \inf_{q \le t \le q+\epsilon} B_t < B_q.$$

The desired property follows.

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Proposition 2.16

Let $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ be a sequence of partitions of $[0, t]$ whose mesh tends to 0 (i.e., $\sup_{1 \le j \le p_n}(t_j^n - t_{j-1}^n) = 0$ as $n \to \infty$). Then

$$\lim_{n \to \infty} \sum_{j=1}^{p_n} (B_{t_j^n} - B_{t_{j-1}^n})^2 = t$$

in L^2.

Proof of Proposition 2.16

This is an immediate consequence of Proposition 1.14, writing $B_{t_j^n} - B_{t_{j-1}^n} = G((t_{j-1}^n, t_j^n])$, where G is the Gaussian white noise associated with B.

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If $a < b$ and f a real-valued function defined on $[a, b]$, the function f is said to have infinite variation if the supremum of $\sum_{j=1}^{p} |f(t_j) - f(t_{j-1})|$, over all partitions $a = t_0 < t_1 < \cdots < t_p = b$, is infinite.

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Proof of Corollary 2.17

From the simple Markov property, it suffices to consider the interval $[0, t]$ for some fixed $t > 0$. Using Proposition 2.16 and extracting a subsequence if necessary, we my assume that the convergence in Proposition 2.16 holds a.s. Now note that

$$\sum_{j=1}^{p_n}(B_{t_j^n} - B_{t_{j-1}^n})^2 \leq \left(\sup_{1\leq j\leq p_n} |B_{t_j^n} - B_{t_{j-1}^n}|\right) \sum_{j=1}^{p_n} |B_{t_j^n} - B_{t_{j-1}^n}|.$$

The lhs tends to 0, the first factor on the rhs also goes to 0, so the 2nd factor on the rhs must go to ∞.

Thus it is impossible to define $\int_0^t f(s)dB_s$ as a pathwise Lebesgue-Stieltjes integral.

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